

# INSTANTON MODULI SPACES ON NON-KÄHLERIAN SURFACES. HOLOMORPHIC MODELS AROUND THE REDUCTION LOCI

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ABSTRACT. Let  $\mathcal{M}$  be a moduli space of polystable rank 2-bundles with fixed determinant (a moduli space of  $\mathrm{PU}(2)$ -instantons) on a Gauduchon surface with  $p_g = 0$  and  $b_1 = 1$ . We study the holomorphic structure of  $\mathcal{M}$  around a circle  $\mathcal{T}$  of regular reductions. Our model space is a "blowup flip passage", which is a manifold with boundary whose boundary is a projective fibration, and whose interior comes with a natural complex structure.

We prove that a neighborhood of the boundary of the blowup  $\hat{\mathcal{M}}_{\mathcal{T}}$  of  $\mathcal{M}$  at  $\mathcal{T}$  can be smoothly identified with a neighborhood of the boundary of a "flip passage"  $\hat{Q}$ , the identification being holomorphic on  $\mathrm{int}(\hat{Q})$ .

## 1. INTRODUCTION

**1.1. Tori of reductions in instanton moduli spaces.** Let  $(M, g)$  be a closed, connected, oriented Riemannian 4-manifold, and  $(E, h)$  be a Hermitian bundle on  $M$ . Let  $a$  be a Hermitian connection on the Hermitian line bundle  $D := \det(E)$ , denote by  $\mathcal{A}(E)$  be the space of Hermitian connections on  $E$ , and put

$$\mathcal{A}_a(E) := \{A \in \mathcal{A}(E) \mid \det(A) = a\}, \quad \mathcal{G}_E := \Gamma(X, \mathrm{SU}(E)), \quad \mathcal{B}_a(E) := \mathcal{A}_a(E) / \mathcal{G}_E$$

$$\mathcal{A}_a^{\mathrm{ASD}}(E) := \{A \in \mathcal{A}_a(E) \mid (F_A^0)^+ = 0\}, \quad \mathcal{M}_a^{\mathrm{ASD}}(E) := \mathcal{A}_a^{\mathrm{ASD}}(E) / \mathcal{G}_E.$$

Using a well known slice theorem (see for instance [DK]), one can prove that the infinite dimensional quotient  $\mathcal{B}_a(E)$ , endowed with the quotient topology, is Hausdorff. Its subspace  $\mathcal{M}_a^{\mathrm{ASD}}(E)$  is finite dimensional and will be called the moduli space of  $a$ -oriented projectively ASD connections on  $(E, h)$ . The open subspace  $\mathcal{B}_a^*(E)$  of  $\mathcal{B}_a(E)$  defined by the condition "A is irreducible" (or equivalently "the stabilizer of A with respect to the  $\mathcal{G}_E$ -action is  $\{\pm \mathrm{id}_E\}$ ") becomes a real analytic Banach manifold after suitable Sobolev completions, and the corresponding subspace  $\mathcal{M}_a^{\mathrm{ASD}}(E)^* \subset \mathcal{M}_a^{\mathrm{ASD}}(E)$  has the structure of a finite dimensional real analytic space. The reduction locus  $\mathcal{M}_a^{\mathrm{ASD}}(E) \setminus \mathcal{M}_a^{\mathrm{ASD}}(E)^*$  can be described as follows.

The set of equivalence classes of decompositions of  $E$  (as orthogonal direct sum of line subbundles) can be identified with the quotient

$$\mathcal{D}ec(E) := \{c \in H^2(X, \mathbb{Z}) \mid c(c_1(E) - c) = c_2(E)\} / \iota$$

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where  $\iota$  is the involution  $c \mapsto c' := c_1(E) - c$ . Suppose  $b_+(M) = 0$  and let  $\lambda = \{c, c'\} \in \mathcal{D}ec(E)$  with  $c \neq c'$ . The moduli subspace

$$\mathcal{T}_\lambda := \{[A] \in \mathcal{M}_a^{\text{ASD}}(E) \mid E \text{ has an } A\text{-parallel line bundle } L \text{ with } c_1(L) \in \lambda\}$$

of  $\mathcal{M}_a^{\text{ASD}}(E)$  is a torus of dimension  $b_1(M)$ , which will be called the torus of  $\lambda$ -reductions. If  $c_1(E) \notin 2H^2(X, \mathbb{Z})$ , then for every  $\lambda = \{c, c'\} \in \mathcal{D}ec(E)$  one has  $c \neq c'$  and any reduction  $A \in \mathcal{A}_a^{\text{ASD}}(E) \setminus \mathcal{A}_a^{\text{ASD}}(E)^*$  will be abelian (has  $S^1$  as stabilizer). In this case one has

$$(1) \quad \mathcal{M}_a^{\text{ASD}}(E) \setminus \mathcal{M}_a^{\text{ASD}}(E)^* = \coprod_{\lambda \in \mathcal{D}ec(E)} \mathcal{T}_\lambda,$$

so the reduction locus of  $\mathcal{M}_a^{\text{ASD}}(E)$  is a disjoint union of tori of dimension  $b_1(M)$ .

Suppose that  $b_+(M) = 0$  and  $\mathcal{T}_\lambda$  is a torus of *regular reductions*, i.e. the second cohomology space of the deformation elliptic complex of  $A$  vanishes for every  $[A] \in \mathcal{T}_\lambda$ . Then one can define the blowup  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  of  $\mathcal{M}_a^{\text{ASD}}(E)$  at the torus  $\mathcal{T}_\lambda$  (see [Te3] section 1.4.2). By construction  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  has the following properties:

- comes with a proper surjective map

$$p_\lambda : \hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda \rightarrow \mathcal{M}_a^{\text{ASD}}(E)$$

which induces an isomorphism  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda \setminus p_\lambda^{-1}(\mathcal{T}_\lambda) \rightarrow \mathcal{M}_a^{\text{ASD}}(E) \setminus \mathcal{T}_\lambda$ ,

- has the structure of a smooth manifold with boundary  $\mathcal{P}_\lambda := p_\lambda^{-1}(\mathcal{T}_\lambda)$  around  $\mathcal{P}_\lambda$ ,
- the induced map  $\pi_\lambda : \mathcal{P}_\lambda \rightarrow \mathcal{T}_\lambda$  is the projectivization of a complex vector bundle over  $\mathcal{T}_\lambda$ .

The existence of the blowup moduli space  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  with the above properties has important consequences:

- (1) Using this result, one can prove easily that a torus of regular reductions has a neighborhood  $N_\lambda$  which is a locally trivial fiber bundle over  $\mathcal{T}_\lambda$  whose fiber is a cone over a complex projective space, and such that  $\mathcal{T}_\lambda$  corresponds to the vertex section of this cone bundle. In other words we have a simple *topological model* of  $\mathcal{M}_a^{\text{ASD}}(E)$  around a torus  $\mathcal{T}_\lambda$  of regular reductions: a cone bundle over  $\mathcal{T}_\lambda$  whose fiber is a cone over a complex projective space.
- (2) Blowing up all tori of reductions  $\mathcal{T}_\lambda$  (which we assume to be regular) in  $\mathcal{M}_a^{\text{ASD}}(E)$  one obtains a blowup moduli space  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)$  on which all Donaldson classes  $c \in H^*(\mathcal{B}_a^*(E), \mathbb{Q})$  extend (see [Te5]). In particular, for any homology class  $h \in H_*(\hat{\mathcal{M}}_a^{\text{ASD}}(E), \mathbb{Q})$  the evaluation  $\langle c, h \rangle$  has sense.

**1.2. Gauduchon stability and the Kobayashi-Hitchin correspondence on non-Kählerian surfaces. The problem.** We refer to [LT1] for the general stability theory on arbitrary compact Gauduchon manifolds. In this article we focus on the situation which is relevant from the point of view of Donaldson theory: rank two bundles on complex surfaces.

Let  $X$  be a complex surface endowed with a *Gauduchon metric*  $g$  [Gau]. A holomorphic rank 2-bundle  $\mathcal{E}$  on  $X$  is called

- *g-stable*, if for every line bundle  $\mathcal{L}$  and sheaf monomorphism  $\mathcal{L} \rightarrow \mathcal{E}$  one has  $\deg(\mathcal{L}) < \frac{1}{2}\deg_g(\det(\mathcal{E}))$ .
- *g-polystable*, if is either stable, or isomorphic to a direct sum  $\mathcal{L} \oplus \mathcal{M}$  of line bundles with  $\deg_g(\mathcal{L}) = \deg_g(\mathcal{M})$ .

Let  $E$  be a differentiable rank 2-bundle on  $X$ ,  $\mathcal{D}$  a fixed holomorphic structure on  $D := \det(E)$ . We denote by  $\mathcal{M}_{\mathcal{D}}^{\text{st}}(E)$ ,  $\mathcal{M}_{\mathcal{D}}^{\text{pst}}(E)$  the moduli space of stable (respectively polystable) holomorphic structures on  $E$  inducing  $\mathcal{D}$  on  $\det(E)$ , modulo the complex gauge group  $\mathcal{G}_E^{\mathbb{C}} := \Gamma(X, \text{SL}(E))$  (see the section 4.2 of the Appendix).  $\mathcal{M}_{\mathcal{D}}^{\text{st}}(E)$  has a natural complex space structure obtained using its open embedding in the corresponding moduli space  $\mathcal{M}_{\mathcal{D}}^{\text{si}}(E)$  of simple holomorphic structures.  $\mathcal{M}_{\mathcal{D}}^{\text{st}}(E)$  is a finite dimensional, but in general non-Hausdorff, complex space [LO]. On the other hand  $\mathcal{M}_{\mathcal{D}}^{\text{pst}}(E)$  is Hausdorff. The Hausdorff property of  $\mathcal{M}_{\mathcal{D}}^{\text{pst}}(E)$  is a consequence of the Kobayashi Hitchin correspondence, which we recall briefly in our framework.

Let  $h$  be a Hermitian metric on  $E$  and let  $a$  be the Chern connection of the pair  $(\mathcal{D}, \det(h))$ . The Kobayashi-Hitchin correspondence states that the map

$$A \mapsto \text{the holomorphic structure defined by } \bar{\partial}_A$$

induces a homeomorphism  $KH : \mathcal{M}_a^{\text{ASD}}(E) \rightarrow \mathcal{M}_{\mathcal{D}}^{\text{pst}}(E)$  which restricts to a real analytic isomorphism  $\mathcal{M}_a^{\text{ASD}}(E)^* \rightarrow \mathcal{M}_{\mathcal{D}}^{\text{st}}(E)$ . More precisely we have a commutative diagram

$$\begin{array}{ccc} \mathcal{B}_a^*(E) \supset \mathcal{M}_a^{\text{ASD}}(E)^* & \hookrightarrow & \mathcal{M}_a^{\text{ASD}}(E) \subset \mathcal{B}_a(E) \\ KH^* \downarrow \simeq & & \simeq \downarrow KH \\ \mathcal{M}_{\mathcal{D}}^{\text{si}}(E) \supset \mathcal{M}_{\mathcal{D}}^{\text{st}}(E) & \hookrightarrow & \mathcal{M}_{\mathcal{D}}^{\text{pst}}(E), \end{array}$$

where  $KH$  is a homeomorphism and  $KH^*$  a real analytic isomorphism.

In general  $\mathcal{M}_{\mathcal{D}}^{\text{pst}}(E)$  is not a complex space around *the reduction locus*  $\mathcal{R} := \mathcal{M}_{\mathcal{D}}^{\text{pst}}(E) \setminus \mathcal{M}_{\mathcal{D}}^{\text{st}}(E)$  ([Te2], [Te3]), which can be identified via  $KH$  with the subspace of reducible instantons in  $\mathcal{M}_a^{\text{ASD}}(E)$ . Under the assumption  $c_1(E) \notin 2H^2(M, \mathbb{Z})$  this subspace has been described above in the general gauge theoretical framework.

Suppose now that  $p_g(X) = 0$  and  $b_1(X) = 1$ . Such a surface is non-Kählerian and has  $b_+(X) = 0$ . In previous articles ([Te2]-[Te4]) we have shown that studying certain moduli spaces of the form  $\mathcal{M}_{\mathcal{D}}^{\text{pst}}(E) = \mathcal{M}_a^{\text{ASD}}(E)$  on such surfaces is interesting, important and difficult. For instance, using a combination of complex geometric and gauge theoretical techniques, we proved geometric properties of certain moduli spaces of this type and we used them to prove existence of curves on class VII surfaces with small  $b_2$ .

An important role in our arguments was played by *the circles of reductions*  $\mathcal{T}_{\lambda}$ . Whereas the blowup construction explained above gives a precise topological description of  $\mathcal{M}_a^{\text{ASD}}(E)$  around a circle  $\mathcal{T}_{\lambda}$  of regular reductions, understanding the complex structure of  $\mathcal{M}_a^{\text{ASD}}(E)^*$  around such a circle is a much more difficult and interesting problem.

The goal of this article is to address this problem, hence to describe explicitly by means of *holomorphic models* the complex structure of the end of the moduli space  $\mathcal{M}_a^{\text{ASD}}(E)^*$  (equivalently of  $\mathcal{M}_{\mathcal{D}}^{\text{st}}(E)$ ) towards a circle  $\mathcal{T}_{\lambda}$  of regular reductions. More precisely we will construct

- an explicit manifold with boundary  $\mathcal{Q}_{\lambda}$  with a complex structure on its interior and whose boundary is a projective bundle over  $\mathcal{T}_{\lambda}$ ,

- a diffeomorphism from  $\mathcal{Q}_\lambda$  onto an open neighborhood  $O_\lambda$  of the boundary  $\mathcal{P}_\lambda$  of  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$ , which induces a biholomorphism  $\text{int}(\mathcal{Q}_\lambda) \rightarrow \text{int}(O_\lambda)$  and a bundle isomorphism  $\partial\mathcal{Q}_\lambda \rightarrow \mathcal{P}_\lambda$  over  $\mathcal{T}_\lambda$ .

Why are we interested in holomorphic models for the ends of  $\mathcal{M}_a^{\text{ASD}}(E)^* = \mathcal{M}_\mathcal{D}^{\text{st}}(E)$ ? The reason is the following problem, which plays an important role in our program for proving existence of curves on class VII surfaces:

**Problem:** Suppose  $c_1(E) \notin 2H^2(X, \mathbb{Z})$ , let  $d = 4c_2(E) - c_1(E)^2$  be the expected complex dimension of  $\mathcal{M}_\mathcal{D}^{\text{st}}(E)$ , and let  $Z \subset \mathcal{M}_\mathcal{D}^{\text{st}}(E)$  be a pure  $k$  dimensional analytic set with  $1 \leq k \leq d$ . Determine explicitly the boundary  $\delta_\lambda([Z]^{BM}) \in H_{2k-1}(\mathcal{P}_\lambda, \mathbb{Z}) \simeq \mathbb{Z}$  of the Borel-Moore fundamental class of  $Z$ .

Intuitively  $\delta_\lambda([Z]^{BM})$  is obtained by intersecting the closure of  $Z$  in  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)$  with the boundary component  $\mathcal{P}_\lambda$ . Suppose that  $\mathcal{M}_a^{\text{ASD}}(E)$  is compact and all reductions in this moduli space are regular. These conditions are satisfied for the moduli spaces studied in [Te2]-[Te3]. Then, for any Donaldson class  $c \in H^{2k-1}(\mathcal{B}_a^*, \mathbb{Q})$  we will have

$$(2) \quad \sum_{\lambda \in \text{Dec}(E)} \langle c|_{\mathcal{P}_\lambda}, \delta_\lambda([Z]^{BM}) \rangle = 0 .$$

The restrictions  $c|_{\mathcal{P}_\lambda}$  have been computed explicitly [Te5]. Therefore, supposing that the problem above has been solved, (2) can be interpreted as an obstruction to the existence of analytic sets  $Z \subset \mathcal{M}_\mathcal{D}^{\text{st}}(E)$  with prescribed topological behavior around the circles of reductions. We will make use of these ideas in a future article.

**1.3. Flip passages and the holomorphic model theorem.** Let  $V', V''$  be Hermitian spaces of dimensions  $r', r''$ . We let  $\mathbb{C}^*$  act on  $V' \times V''$  by

$$\zeta \cdot (y', y'') := (\zeta y', \zeta^{-1} y'') .$$

The induced  $S^1$ -action has a 1-parameter family  $(\mu_t)_{t \in \mathbb{R}}$  of moment maps given by  $\mu_t = -im_t$ , where  $m_t(y', y'') = \frac{1}{2}(\|y'\|^2 - \|y''\|^2) + t$ . The corresponding family of Kähler quotients is:

$$Q_t := m_t^{-1}(0)/_{S^1} = \begin{cases} Q' := V'_* \times V''/\mathbb{C}^* & \text{for } t < 0 \\ Q_0 := \{(V'_* \times V''_*) \cup \{0\}\}/\mathbb{C}^* & \text{for } t = 0 \\ Q'' := V' \times V''_*/\mathbb{C}^* & \text{for } t > 0 \end{cases} .$$

Denoting by  $\Theta', \Theta''$  be the tautological line bundles over  $\mathbb{P}(E'), \mathbb{P}(E'')$  we get obvious biholomorphisms:

$$Q' \stackrel{\text{bihol}}{\simeq} \{\Theta' \otimes V''\} \stackrel{\text{bihol}}{\simeq} \{\Theta'\}^{\oplus r''} , \quad Q'' \stackrel{\text{bihol}}{\simeq} \{\Theta'' \otimes V'\} \stackrel{\text{bihol}}{\simeq} \{\Theta''\}^{\oplus r'}$$

$Q_0$  can be identified with the image of the natural map  $V' \times V'' \rightarrow V' \otimes V''$ , and (if  $r' > 0, r'' > 0$ ) it has an isolated singularity. It will be called *the singular quotient* and can be identified with the cone (in the algebraic geometric sense) over the image of the Segre embedding  $\mathbb{P}(V') \times \mathbb{P}(V'') \rightarrow \mathbb{P}(V' \otimes V'')$ . We also define *the blowup quotient*  $\tilde{Q}$  by

$$\tilde{Q} := p'^*(\Theta') \otimes p''^*(\Theta'') ,$$

which is a holomorphic line bundle over  $\mathbb{P}(V') \times \mathbb{P}(V'')$ . These spaces fit in the commutative diagram

$$\begin{array}{ccccc}
 & & \mathbb{P}(V') \times \mathbb{P}(V'') & & \\
 & & \downarrow & & \\
 & & \tilde{Q} & & \\
 & \swarrow & & \searrow & \\
 \mathbb{P}(V') \subset Q' & \xleftarrow{\quad} & \text{a flip} & \xrightarrow{\quad} & Q'' \supset \mathbb{P}(V'') \\
 & \searrow & & \swarrow & \\
 & & Q_0 & & \\
 & & \uparrow & & \\
 & & \{*\} & & 
 \end{array}$$

in which the four maps define biholomorphisms between the complements of the corresponding closed subspaces appearing in the diagram. The birational map  $Q' \dashrightarrow Q''$  is a standard example of a flip (see [R]).

Let  $p' : E' \rightarrow B$ ,  $p'' : E'' \rightarrow B$  be holomorphic bundles of ranks  $r'$ ,  $r''$  endowed with Hermitian metrics  $h'$ ,  $h''$  on a complex manifold  $B$ . Endow  $E := E' \oplus E''$  with the  $\mathbb{C}^*$ -action

$$\zeta \cdot (y', y'') = (\zeta y', \zeta^{-1} y'') .$$

Let  $f : B \rightarrow \mathbb{R}$  a smooth map which is a submersion at any vanishing point. Therefore the zero set  $T := Z(f)$  is an oriented real hypersurface of  $B$ . We assume that  $T$  is compact. We define the smooth family of fiberwise moment maps associated with  $f$  by  $\mu^f = -im^f : E \rightarrow i\mathbb{R}$ , where  $m^f$  is given fiberwise by

$$m_b^f(y', y'') = \frac{1}{2}(\|y'\|^2 - \|y''\|^2) + f(b) , \quad \forall (y', y'') \in E_b .$$

Put

$$Q_f := Z(f)/S^1 , \quad Q_f^* := Z(m) \setminus Z(m)^{S^1}/S^1 .$$

One has an obvious identification (induced by  $p$ )  $\{Z(m)\}^{S^1} = T$ .  $Q_f^*$  comes with an obvious open embedding  $Q_f^* \hookrightarrow \{E \setminus B\}/\mathbb{C}^*$ , hence is naturally a complex manifold of dimension  $r' + r'' + \dim(B) - 1$ . Therefore  $Q_f$  is obtained by adding the real hypersurface  $T$  to the complex manifold  $Q_f^*$ . Note that  $Q_f$  comes with a map  $Q_f \rightarrow B$  whose fibers are

$$\{Q_f\}_b \simeq \begin{cases} Q' & \text{for } f(b) < 0 \\ Q_0 & \text{for } f(b) = 0 \\ Q'' & \text{for } f(b) > 0 \end{cases} .$$

Therefore  $Q_f$  should be called the *flip passage* from  $Q'$  to  $Q''$  (passing through  $Q_0$ ) associated with the system  $(B, E', E'', h', h'', f)$ . The complement  $T = Q_f \setminus Q_f^*$  is formed by the singularities of the fibers of type  $Q_0$ . The *blowup flip passage*  $\hat{Q}_f$  is defined by  $\hat{Q}_f := \widehat{m_f^{-1}(0)}/S^1$ , where  $\widehat{m_f^{-1}(0)}$  is the spherical blowup of the smooth hypersurface  $m_f^{-1}(0) \subset E$  at the  $S^1$ -fixed point locus.  $\hat{Q}_f$  is a manifold with boundary, whose interior coincides with the complex manifold  $Q_f^*$  and whose boundary  $\partial\hat{Q}_f$  can be identified with the projective bundle  $\mathbb{P}(E'|_T \oplus \bar{E}''|_T)$  (see

section 2.1 for details).

Let now  $(X, g)$  be a Gauduchon surface with  $p_g = 0$ ,  $b_1(X) = 1$ . For such a surface one has  $\text{Pic}^0(X) \simeq \mathbb{C}^*$  and, choosing an isomorphism  $\mathbb{C}^* \ni \zeta \mapsto \mathcal{L}_\zeta \in \text{Pic}^0(X)$  in a convenient way, one has

$$\deg_g(\mathcal{L}_\zeta) = C_g \ln |\zeta| ,$$

for a positive constant  $C_g$  depending smoothly on  $g$ . This shows that the level sets of the restriction of  $\deg_g$  on any component of  $\text{Pic}(X)$  are circles.

Let  $(E, h)$  be a Hermitian rank 2-bundle on  $X$ ,  $\mathcal{D}$  a fixed holomorphic structure on  $D := \det(E)$  and  $a$  the Chern connection of the pair  $(\mathcal{D}, \det(h))$ . Let  $L$  be a line subbundle of  $E$  with  $2c_1(L) \neq c_1(E)$  and  $\lambda = \{c_1(L), c_1(E) - c_1(L)\}$  the corresponding element of  $\mathcal{D}ec(E)$ . Suppose that  $\mathcal{T}_\lambda \subset \mathcal{M}_a^{\text{ASD}}(E)$  is a circle of *regular* reductions.

Fix  $x_0 \in X$  and let  $\mathcal{L} = \mathcal{L}_{x_0}$  be the Poincaré line bundle associated with the base point  $x_0$  (see Definition 4.7, section 4.3) on  $\text{Pic}^c(X) \times X$  endowed with its canonical Hermitian metric (see Remark 4.9). Denoting by  $p_1 : \text{Pic}^c(X) \times X \rightarrow \text{Pic}^c(X)$ ,  $p_2 : \text{Pic}^c(X) \times X \rightarrow X$  the two projections, put

$$\mathcal{H}' := R^1(p_1)_*(\mathcal{L}^{\otimes 2} \otimes p_2^*(\mathcal{D}^\vee)) , \quad \mathcal{H}'' := R^1(p_1)_*(\mathcal{L}^{-\otimes 2} \otimes p_2^*(\mathcal{D})) ,$$

and let  $f_{\mathcal{D}} : \text{Pic}^c(X) \rightarrow \mathbb{R}$  be the harmonic map defined by

$$f_{\mathcal{D}}([\mathcal{L}]) := \pi(\deg_g(\mathcal{L}) - \frac{1}{2}\deg_g(\mathcal{D})) ,$$

The vanishing circle  $T := Z(f_{\mathcal{D}})$  can be identified with the reduction torus  $\mathcal{T}_\lambda \subset \mathcal{M}_{\mathcal{D}}^{\text{pst}}(E)$  using the map  $k : T \rightarrow \mathcal{T}_\lambda$  given by

$$k_\lambda(\mathcal{L}) := [\mathcal{L} \oplus (\mathcal{D} \otimes \mathcal{L}^\vee)] .$$

For  $\varepsilon > 0$  consider the annulus  $\text{Pic}_\varepsilon^c(X) := f_{\mathcal{D}}^{-1}(-\varepsilon, \varepsilon)$ . Under our assumptions it follows that, for any sufficiently small  $\varepsilon > 0$ , the restrictions

$$\mathcal{H}'|_{\text{Pic}_\varepsilon^c(X)} , \quad \mathcal{H}''|_{\text{Pic}_\varepsilon^c(X)}$$

are locally free of ranks

$$r' = -\frac{1}{2}(2c - c_1(E))(2c - c_1(E) + c_1(X)) , \quad r'' = -\frac{1}{2}(-2c + c_1(E))(-2c + c_1(E) + c_1(X))$$

respectively, and for any  $l \in \text{Pic}_\varepsilon^c(X)$  one has canonical identifications

$$\mathcal{H}'(l) = H^1(\mathcal{L}_l^{\otimes 2} \otimes p_2^*(\mathcal{D}^\vee)) , \quad \mathcal{H}''(l) = H^1(\mathcal{L}_l^{\otimes -2} \otimes p_2^*(\mathcal{D})) .$$

We will see that, for  $l \in T$ , these spaces come with natural Hermitian products obtained by identifying them with suitable harmonic spaces (see Proposition 4.2), which will be endowed with standard  $L^2$ -Hermitian products. In this way we get Hermitian metrics  $\mathfrak{h}'$ ,  $\mathfrak{h}''$  on the bundles  $\mathcal{H}'|_T$ ,  $\mathcal{H}''|_T$ , which can be extended to get Hermitian metrics  $h'$ ,  $h''$  on  $\mathcal{H}'|_{\text{Pic}_\varepsilon^c(X)}$ ,  $\mathcal{H}''|_{\text{Pic}_\varepsilon^c(X)}$ . Let  $Q$  ( $\hat{Q}$ ) be the (blowup) flip passage associated with the system

$$(\text{Pic}_\varepsilon^c(X), \mathcal{H}'|_{\text{Pic}_\varepsilon^c(X)}, \mathcal{H}''|_{\text{Pic}_\varepsilon^c(X)}, h', h'', f_{\mathcal{D}}|_{\text{Pic}_\varepsilon^c(X)}) .$$

Our holomorphic model theorem states:

**Theorem 1.1.** *Under the assumptions and with the notations above there exists an open neighborhood  $O$  of  $\partial\hat{Q}$  in  $\hat{Q}$  and a diffeomorphism  $\chi : O \rightarrow \mathcal{O}_\lambda$  onto a smooth open neighborhood  $\mathcal{O}_\lambda$  of  $\mathcal{P}_\lambda$  in the blow up moduli space  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  such that*

1.  $\chi$  induces a smooth bundle isomorphism

$$(3) \quad \begin{array}{ccc} \mathbb{P}(\mathcal{H}'|_T \oplus \bar{\mathcal{H}}''|_T) = \partial\hat{Q} & \xrightarrow{\partial\chi} & \mathcal{P}_\lambda \\ \downarrow & & \downarrow \pi_\lambda \\ T & \xrightarrow{k_\lambda} & \mathcal{T}_\lambda \end{array},$$

2.  $\chi$  induces a biholomorphism  $O \setminus \partial\hat{Q} \rightarrow \mathcal{O}_\lambda \setminus \mathcal{P}_\lambda$ .

Therefore, around the boundary  $\mathcal{P}_\lambda$ , the blowup moduli space  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  can be identified with a neighborhood of the boundary of the blowup flip passage associated with the system  $(\text{Pic}_\varepsilon^c(X), \mathcal{H}'|_{\text{Pic}_\varepsilon^c(X)}, \mathcal{H}''|_{\text{Pic}_\varepsilon^c(X)}, h', h'', f_{\mathcal{D}}|_{\text{Pic}_\varepsilon^c(X)})$ , and this identification respects the complex structure on  $\mathcal{M}_a^{\text{ASD}}(E)^*$  induced by the Kobayashi-Hitchin correspondence. The difficult part of the result is the holomorphy property of our model. Note that the holomorphic bundles  $\mathcal{H}'|_{\text{Pic}_\varepsilon^c(X)}, \mathcal{H}''|_{\text{Pic}_\varepsilon^c(X)}$  are in fact trivial.

Collapsing to points the fibers of the two projective fibrations in (3), we obtain the following *weaker* holomorphic model theorem:

**Corollary 1.2.** *There exists a homeomorphism  $U \rightarrow U_\lambda$  from an open neighborhood  $U$  of  $T$  in  $Q_f$  onto an open neighborhood  $U_\lambda$  of  $\mathcal{T}_\lambda$  in  $\mathcal{M}_a^{\text{ASD}}(E)$ , which restricts to the standard identification  $k_\lambda : T \rightarrow \mathcal{T}_\lambda$  and a biholomorphism  $U \setminus T \rightarrow U_\lambda \setminus \mathcal{T}_\lambda$ .*

## 2. $\mathbb{C}^*$ -QUOTIENTS OF HOLOMORPHIC BUNDLES WITH RESPECT TO FAMILIES OF MOMENT MAPS

**2.1. Blowup  $S^1$ -quotients and families of  $S^1$ -moment maps on Hermitian bundles.** Let  $Z$  be a differentiable manifold endowed with an  $S^1$ -action such that the stabilizer of any point  $z \in Z$  is either  $S^1$  or trivial. This implies that the fixed point locus  $Z^{S^1}$  is a submanifold of  $Z$ , and that the normal bundle  $N_{Z^{S^1}}^Z$  of  $Z^{S^1}$  in  $Z$  has a complex structure such that the induced action of  $S^1$  is the standard one. The induced  $S^1$ -action on the spherical blowup [AK]  $\hat{Z}_{Z^{S^1}}$  of  $Z$  at  $Z^{S^1}$  is free, and the corresponding quotient has a natural structure of a manifold with boundary, whose boundary can be identified with  $\mathbb{P}(N_{Z^{S^1}}^Z)$ .

**Definition 2.1.** *The quotient  $\hat{Z}_{Z^{S^1}}/S^1$ , endowed with its natural structure of a manifold with boundary, will be called the blowup  $S^1$ -quotient of  $Z$  and will be denoted by  $\hat{Z}/S^1$ .*

Let  $p' : E' \rightarrow B$ ,  $p'' : E'' \rightarrow B$  be the projection maps of two holomorphic vector bundles over a (connected) complex manifold  $B$ , and  $p : E := E' \times_B E'' \rightarrow B$  the projection map of their direct sum  $E' \oplus E''$ . We let  $\mathbb{C}^*$  act on  $E$  by

$$\zeta(u', u'') := (\zeta u', \zeta^{-1} u'').$$

Identify in the obvious way the base  $B$  with the image  $\{(0'_b, 0''_b) \mid b \in B\} \subset E$  of the zero section of  $E$ .

**Remark 2.2.** *The induced  $\mathbb{C}^*$ -action on  $E^* := E \setminus B$  is free. The quotient  $E^*/\mathbb{C}^*$  has the structure of a complex manifold which makes the canonical projection*

$$E^* \rightarrow E^*/\mathbb{C}^*$$

*a holomorphic submersion. This manifold is non-Hausdorff if  $r' > 0$  and  $r'' > 0$ .*

We can obtain a Hausdorff  $\mathbb{C}^*$ -quotient using ideas from the theory of Kählerian quotients (see for instance [Te1]). We will not use a moment map for the induced  $S^1$ -action on the total space  $E$ , but a family of fiberwise moment maps parameterized by the base  $B$ . Fix Hermitian metrics  $h'$ ,  $h''$  on  $E'$ ,  $E''$  respectively, and endow every fiber  $E_b = E'_b \times E''_b$  with the corresponding product Kähler metric.

**Definition 2.3.** Let be  $f : B \rightarrow \mathbb{R}$  a smooth map. The family of fiberwise moment maps associated with  $f$  is the smooth map  $\mu^f = -im^f : E \rightarrow i\mathbb{R}$ , where  $m^f$  is given fiberwise by

$$m_b^f(y', y'') = \frac{1}{2}(\|y'\|^2 - \|y''\|^2) + f(b), \quad \forall (y', y'') \in E_b.$$

The fiberwise stable locus associated with  $\mu^f$  is the open set

$$E_f^{\text{st}} := \left\{ y = (y', y'') \in E \left| \begin{cases} y' \neq 0 & \text{if } f(p(y)) < 0 \\ y'' \neq 0 & \text{if } f(p(y)) > 0 \\ y' \neq 0 \text{ and } y'' \neq 0 & \text{if } f(p(y)) = 0 \end{cases} \right. \right\} \subset E,$$

and the quotient  $E_f^{\text{st}}/\mathbb{C}^*$  is a smooth complex manifold of dimension  $\dim(B) + r' + r'' - 1$ . This quotient is Hausdorff because, as explained in section 1.3, it can be identified with the open subspace  $Q_f^* := \{Z(m^f) \setminus Z(m^f)^{S^1}\}/S^1$  of the  $S^1$ -quotient

$$Q_f := Z(m^f)/_{S^1}$$

which is Hausdorff, as the quotient of a Hausdorff space by a compact group.

Recall that in the theory of Kähler quotients one is given a holomorphic action of a complex reductive group  $G$  on a Kähler manifold whose Kähler structure is invariant under a maximal compact subgroup  $K \subset G$ , and a moment map  $\mu$  for the induced  $K$ -action. In this framework the  $K$ -quotient  $Z(\mu)/K$  has the structure of a complex space which can be identified with the *good*  $G$ -quotient of the semistable locus (see [Te1]). The analogous property does not hold in our situation, hence the complex structure of  $Q_f^*$  might not extend to  $Q_f = Z(m^f)/S^1$ . This happens because the map  $\mu^f = -im^f$  is not a moment map for the  $S^1$ -action on  $E$ , but just a smooth family of fiberwise moment maps. The complement  $Q_f \setminus Q_f^*$  can be identified with the fixed point locus  $Z(m^f)^{S^1}$ , which can be further identified with the zero locus  $Z(f) \subset B$ .

Suppose that  $f$  is a submersion at any vanishing point. This implies that the zero locus  $T := Z(f)$  is a smooth, *oriented* real hypersurface of  $B$ .  $T$  is oriented in the standard way. With this convention, the oriented manifold  $\pm T$  is the boundary of the oriented manifold with boundary  $T_{\pm} := (\pm f)^{-1}([0, \infty))$ .

Moreover this also implies  $m^f$  is a submersion at any vanishing point, so that  $Z(m^f)$  becomes a smooth smooth, oriented real hypersurface of  $E$  which is endowed with an  $S^1$ -action and intersects the zero section  $B$  transversally along the fixed point locus  $Z(f)$ . This  $S^1$ -action is free away of this fixed point locus. Therefore, we are precisely in the situation appearing in Definition 2.1, in which we introduced the concept of a blowup  $S^1$ -quotient.

**Definition 2.4.** Suppose that  $f$  is a submersion at any vanishing point and that the real hypersurface  $T = Z(f)$  is compact. The (blowup) flip passage associated with the system  $(B, E', E'', h', h'', f)$  is the quotient  $Q_f := Z(m^f)/S^1$  (respectively the blowup  $S^1$ -quotient  $\hat{Q}_f := \widehat{Z(m^f)}/S^1$ ).



Therefore, by definition, the blowup flip passage  $\hat{Q}_f$  is a manifold with boundary

$$\partial\hat{Q}_f = \mathbb{P}(N_T^{Z(m^f)}) ,$$

and its interior can be identified with  $Q_f^*$ , hence it comes with a natural complex structure. Putting

$$F' := E'|_T , \quad F'' := E''|_T ,$$

we have an isomorphism of  $S^1$ -bundles over  $T$

$$T_{Z(m^f)}|_T = T_T \oplus (F' \oplus \bar{F}'') , \quad N_T^{Z(m^f)} = F' \oplus \bar{F}'' , \quad \partial\hat{Q}_f = \mathbb{P}(F' \oplus \bar{F}'') .$$

Note that the boundary orientation of  $\partial\hat{Q}_f$  (with respect to the complex orientation of the interior of  $\hat{Q}_f$ ) does not coincide in general with the natural orientation of the projective fibration  $\mathbb{P}(F' \oplus \bar{F}'')$  over the oriented manifold  $T$ . The two orientations can be compared easily.

**2.2. Perturbations of  $m^f$ .** Let  $(B, p' : E' \rightarrow B, p'' : E'' \rightarrow B, h', h'', f)$  be a system as above, where  $f : B \rightarrow \mathbb{R}$  is a submersion at any point of the zero locus  $T := Z(f)$ , which we assume to be compact. Put

$$F' := E'|_T , \quad F'' := E''|_T .$$

Identifying as usual  $B$  with the zero section of  $E$  we have a natural identification of  $S^1$ -bundles

$$T_E|_B = T_B \oplus E' \oplus \bar{E}'' .$$

Let  $U \subset E$  be an  $S^1$ -invariant open neighborhood of  $T \subset E$  and  $\varphi : U \rightarrow \mathbb{R}$  a smooth,  $S^1$ -invariant map with the following properties:

- P1.  $Z(\varphi|_B) = T$ , and  $d_x(\varphi|_B) = d_x f$  for any  $x \in T$ .
- P2. For every  $x \in T$  one has

$$D_x \varphi|_{E'_x} = 0 , \quad D_x \varphi|_{E''_x} = 0 ,$$

- P3. For every  $x \in T$  the point  $x$  is a non-degenerate critical point of the fiberwise restriction  $\varphi_x := \varphi|_{E_x \cap U}$ , and the second derivative at  $x$  of this restriction is

$$D_x^2(\varphi_x)((u'_1, u''_1), (u'_2, u''_2)) = \Re(h'_x(u'_1, u'_2)) - \Re(h''_x(u''_1, u''_2)) .$$

Using conditions P1, P2 we obtain isomorphisms of  $S^1$ -bundles over  $T$

$$T_{Z(\varphi)}|_T = T_T \oplus F' \oplus \bar{F}'' , \quad N_T^{Z(\varphi)} = F' \oplus \bar{F}'' ,$$

hence  $\hat{Q}_f$  and the blowup  $S^1$ -quotient of  $Z(\varphi)$  have the same boundary. The following proposition shows that, replacing  $m^f$  by  $\varphi$  in the definition of  $\hat{Q}_f$  one obtains the same *complex* manifold with boundary around this common boundary.

**Proposition 2.5.** *Let  $(B, p' : E' \rightarrow B, p'' : E'' \rightarrow B, h', h'', f)$  be a system as above, where  $f : B \rightarrow \mathbb{R}$  is a submersion at any point of  $T := Z(f)$ , which we assume to be compact. Let  $\varphi : U \rightarrow \mathbb{R}$  be a smooth,  $S^1$ -invariant map satisfying the conditions P1 - P3 above. There exists an  $S^1$ -invariant, open neighborhood  $V$  of  $T$  in  $U$  such that*

1.  $\varphi$  is a submersion at any vanishing point of  $\varphi|_V$ , in particular the zero locus  $Z_{V, \varphi} := Z(\varphi|_V)$  is a smooth real hypersurface of  $V$ .
2. Denoting  $Z_{V, \varphi}^* := Z_V^\varphi \setminus B$  one has  $Z_{V, \varphi}^* \subset E_f^{\text{st}}$ .

3. The map  $Q_{V,\varphi}^* := Z_{V,\varphi}^*/S^1 \rightarrow E_f^{\text{st}}/\mathbb{C}^* = Q_f^*$  induced by the inclusion  $Z_{V,\varphi}^* \hookrightarrow E_f^{\text{st}}$  has the properties:

(a) extends to a smooth open embedding of manifolds with boundary

$$\hat{Q}_{V,\varphi} := \widehat{Z_{V,\varphi}}/S^1 \hookrightarrow \widehat{Z(m^f)}/S^1 = \hat{Q}_f$$

which induces the identity map between the boundaries (via the identifications  $\partial\hat{Q}_{V,\varphi} = \mathbb{P}(F' \oplus \bar{F}'') = \partial\hat{Q}_f$ ).

(b) is an open embedding which becomes holomorphic if we endow  $Q_{V,\varphi}^*$  with the complex structure induced from  $E_f^{\text{st}}/\mathbb{C}^*$ .

*Proof.* Note first that a sufficiently small tubular neighborhood of  $T$  in  $B$  can be smoothly identified with  $(-\varepsilon, \varepsilon) \times T$  such that  $f$  is given by the projection on the first factor. Furthermore, using Hermitian connections on  $E'$ ,  $E''$  and parallel transport along the curves  $(-\varepsilon, \varepsilon)$ , one can identify these Hermitian bundles with  $E' = F' \times (-\varepsilon, \varepsilon)$ ,  $E'' = F'' \times (-\varepsilon, \varepsilon)$  where  $F' := E'|_T$ ,  $F'' := E''|_T$ . Therefore, since our problem is local with respect to  $T$ , we can suppose that

- $B = T \times (-\varepsilon, \varepsilon)$  (as real differentiable manifolds) and  $f : T \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  is given by the projection on the second factor.
- $E' = F' \times (-\varepsilon, \varepsilon)$ ,  $E'' = F'' \times (-\varepsilon, \varepsilon)$ , where  $q' : F' \rightarrow T$ ,  $q'' : F'' \rightarrow T$  are Hermitian bundles on  $T$ .

1. By P1 the map  $\varphi$  is a submersion at any point  $x \in T$ , hence it is a submersion on an open neighborhood  $V$  of  $T$ . Since  $S^1$  is compact we may suppose that  $V$  is  $S^1$ -invariant.

2. Put  $F := F' \times_T F''$ . Using P2 we see that the hypersurface  $Z_{V,\varphi} \subset V$  is vertical along  $T$ , i.e, the restriction of its tangent bundle to  $T$  coincides with  $F$ . Therefore, we may suppose that, for sufficiently small, relatively compact,  $S^1$ -invariant open neighborhoods  $V'$ ,  $V''$  of  $T$  in  $F'$  and  $F''$ , the following holds:

- $V = (V' \times_T V'') \times (-\varepsilon, \varepsilon)$ ,
- $Z_{V,\varphi}$  is the graph of an  $S^1$ -invariant function  $\chi : V' \times_T V'' \rightarrow (-\varepsilon, \varepsilon)$  which vanishes on  $T$ , and whose differential vanishes at any point of  $T$ .

In other words the first order jet of  $\chi$  along  $T$  vanishes. The equality  $Z_{V,\varphi} = \text{graph}(\chi)$  implies  $\varphi(y, \chi(y)) = 0$  for any  $y \in V' \times_T V''$ . Differentiating twice this identity at a point  $x \in T$  in fiber directions, and taking into account P1 and P3 we obtain

$$\begin{aligned} D_x^2 \chi((u'_1, u''_1), (u'_2, u''_2)) &= -D_x^2(\varphi_x)((u'_1, u''_1), (u'_2, u''_2)) \\ &= -\Re(h'_x(u'_1, u'_2)) + \Re(h''_x(u''_1, u''_2)) . \end{aligned}$$

The order 2-Taylor development of the fiber restriction  $\chi_x := \chi|_{F_x}$  reads

$$(4) \quad \chi_x(u', u'') = -\frac{1}{2}(h'_x(u', u') - h''_x(u'', u'')) + r_x(u', u'') ,$$

where, putting  $u = (u', u'')$  the rest  $r_x$  is given by the integral formula

$$(5) \quad r_x(u) = \int_0^1 \frac{(1-t)^2}{2} D_{tu}^3(\chi_x)(u, u, u) dt .$$

Since we supposed  $V'$ ,  $V''$  to be relatively compact, we get a uniform bound

$$(6) \quad |r_x(u', u'')| \leq M \{h'_x(u', u') + h''_x(u'', u'')\}^{\frac{3}{2}} .$$

Taking  $V'$ ,  $V''$  sufficiently small we will have

$$(7) \quad |r_x(u', 0)| \leq \frac{h'_x(u', u')}{4}, \quad |r_x(0, u'')| \leq \frac{h''_x(u'', u'')}{4}, \quad \forall x \in T \quad \forall (u', u'') \in V'_x \times V''_x.$$

On the other hand, letting  $x$  vary in  $T$ , the hypersurface

$$Z_{V,\varphi} = \text{graph}(\chi) \subset (V' \times_T V'') \times (-\varepsilon, \varepsilon)$$

is defined by the equation

$$t = -\frac{1}{2}(h'(u', u') - h''(u'', u'')) + r(u', u''),$$

hence, taking into account (7), we obtain

$$(u', 0, t) \in Z_{V,\varphi}^* \Rightarrow t < 0, \quad (0, u'', t) \in Z_{V,\varphi}^* \Rightarrow t > 0,$$

hence  $Z_{V,\varphi}^* \subset E_f^{\text{st}}$  as claimed.

3. We have to compare the quotients  $\hat{Q}_{V,\varphi} = \widehat{Z_{V,\varphi}}/S^1$ ,  $\hat{Q}_f = \widehat{Z(m^f)}/S^1$ . We will first compare the two hypersurfaces

$$Z_{V,\varphi} = \text{graph}(\chi), \quad Z(m^f) = \text{graph}(\chi_f),$$

where  $\chi : V' \times_T V'' \rightarrow \mathbb{R}$  has been defined above and  $\chi_f : F' \times_T F'' \rightarrow \mathbb{R}$  is given by

$$\chi_f(v', v'') := -\frac{1}{2}(h'_x(v', v') - h''_x(v'', v'')).$$

The maps  $g_f : F = F' \times_T F'' \rightarrow Z(m^f)$ ,  $g : V' \times_T V'' \rightarrow Z_{V,\varphi}$  given by

$$g_f(v', v'') := (v', v'', \chi_f(v', v'')), \quad g(v', v'') := (v', v'', \chi(v', v''))$$

are  $S^1$ -equivariant diffeomorphisms, hence they induce  $S^1$ -equivariant diffeomorphisms

$$\hat{g}_f : \hat{F} \rightarrow \hat{Z}(m^f), \quad \hat{g} : \widehat{V' \times_T V''} \rightarrow \hat{Z}_{V,\varphi}$$

between the corresponding spherical blowups. Note now that the  $\mathbb{C}^*$ -action on  $F$  given by  $(\zeta, v', v'') \mapsto (\zeta v', \zeta^{-1} v'')$  extends to a  $\mathbb{C}^*$ -action on the spherical blowup  $\hat{F}$ , which is given by

$$\zeta \cdot (r, w) = \left( r(|\zeta|^2 \|w'\|^2 + |\zeta|^{-2} \|w''\|^2)^{\frac{1}{2}}, \frac{1}{(|\zeta|^2 \|w'\|^2 + |\zeta|^{-2} \|w''\|^2)^{\frac{1}{2}}} (\zeta w', \zeta^{-1} w'') \right)$$

and leaves invariant  $\partial \hat{F}$ . Define  $U : \widehat{V' \times_T V''} \rightarrow \widehat{F' \times_T F''}$  by

$$U(y) = \rho(y) \cdot y,$$

where  $\rho : \widehat{V' \times_T V''} \rightarrow \mathbb{R}_{>0}$  is the smooth function given by Lemma 2.6 below. This map acts as the identity on the boundary  $\widehat{V' \times_T V''}$  and on its normal line bundle. The composition  $\mathfrak{U} := \hat{g}_f \circ U \circ \hat{g}^{-1}$  is a smooth,  $S^1$ -equivariant map  $\hat{Z}_{V,\varphi} \rightarrow \hat{Z}(m^f)$  which induces  $\text{id}_{S(F)}$  on the common boundary, and is a local diffeomorphism at any point of this boundary. Therefore, passing to  $S^1$ -quotients,  $\mathfrak{U}$  induces a smooth map  $\mathfrak{u} = \hat{Q}_{V,\varphi} \rightarrow \hat{Q}_f$  which acts as identity on  $\partial \hat{Q}_{V,\varphi}$  and is a local diffeomorphism at any point of  $\partial \hat{Q}_{V,\varphi}$ . Applying the inverse function theorem at the boundary points and replacing  $V$  by a smaller  $S^1$ -invariant open neighborhood of  $T$  if necessary,  $\mathfrak{u}$  will become a smooth open embedding. Moreover, using (8), we see that for a point  $y = (v', v'', t) \in Z_{V,\varphi}^*$  one has

$$\mathfrak{U}(y) = (\rho(y)v', \rho(y)^{-1}v'', t) \in \mathbb{C}^*y,$$

which shows that  $u$  extends the natural map  $Q_{V,\varphi}^* \rightarrow Q_f^*$  induced by the inclusion  $Z_{V,\varphi}^* \hookrightarrow E_f^{\text{st}}$ . This proves claim 3a. Claim 3b is an obvious consequence of 3a.  $\blacksquare$

**Lemma 2.6.** *Let  $V'$ ,  $V''$  be open,  $S^1$ -invariant open neighborhoods of the zero sections in  $F'$ ,  $F''$  as in the proof of conclusions 1., 2. of Proposition 2.5. The equation*

$$(8) \quad \chi_f(\rho v', \rho^{-1} v'') = \chi(v', v'')$$

*has a unique solution for every  $(v', v'') \in (V' \times_T V'') \setminus T$  and the obtained function  $(V' \times_T V'') \setminus T \rightarrow (0, \infty)$  extends to a smooth,  $S^1$ -invariant function*

$$\rho : \widehat{V' \times_T V''} \rightarrow (0, \infty)$$

*with  $\rho|_{S(F' \oplus F'')} \equiv 1$ .*

*Proof.* It's easy to see that (8) has a unique solution in  $(0, \infty)$  when  $v = (v', v'')$  does not belong to the zero section  $T \subset F$ . Indeed, for  $v = (v', v'') \neq 0$ , the equation  $\chi_f(\rho v', \rho^{-1} v'') = c$  has always a unique positive solution except when  $v'' = 0$  and  $c > 0$  and when  $v' = 0$  and  $c < 0$ . But conclusion 2. of Proposition 2.5 shows that  $\chi(v', 0) < 0$  when  $v' \neq 0$  and  $\chi(0, v'') > 0$  when  $v'' \neq 0$ .

The obtained map  $(V' \times_T V'') \setminus T \rightarrow (0, \infty)$  is  $S^1$ -invariant because  $\chi_f$  and  $\chi$  are  $S^1$ -invariant. We have to prove that this function extends to a smooth map  $\rho : \widehat{V' \times_F V''} \rightarrow (0, \infty)$  satisfying  $\rho|_{S(F)} \equiv 1$ . Applying Lemma 2.7 below to the functions  $\alpha, \beta : (0, \infty) \times V' \times_T V'' \rightarrow \mathbb{R}$  given by

$$\alpha(\rho, v', v'') = \chi_f(\rho v', \rho^{-1} v'') - \chi(v', v'') ,$$

$$\beta(\rho, v', v'') = \frac{\partial}{\partial \rho} \alpha(\rho, v', v'') = -\rho^{-1} (\rho^2 h'(v', v') + \rho^{-2} h''(v'', v''))$$

we obtain two smooth function  $\hat{\alpha}, \hat{\beta} : (0, \infty) \times \widehat{V' \times_F V''} \rightarrow \mathbb{R}$  extending the functions

$$(\rho, v', v'') \mapsto \frac{1}{\|v\|^2} \alpha(\rho, v', v'') , \quad (\rho, v', v'') \mapsto \frac{1}{\|v\|^2} \beta(\rho, v', v'') ,$$

and whose restrictions to the boundary  $(0, \infty) \times S(F)$  are given by

$$(9) \quad \hat{\alpha}(\rho, w', w'') = -\frac{\rho^2 - 1}{2} \left( h'(w', w') + \frac{1}{\rho^2} h''(w'', w'') \right)$$

$$(10) \quad \hat{\beta}(\rho, w', w'') = -\rho^{-1} (\rho^2 h'(w', w') + \rho^{-2} h''(w'', w'')) .$$

For (9) we used (4) and (6) to compute  $\lim_{r \rightarrow 0} \frac{1}{r^2} \chi(rw)$ . Since  $\frac{\partial}{\partial \rho} \hat{\alpha} = \hat{\beta}$  away of the boundary  $(0, \infty) \times S(F)$ , it follows that this equality holds on the whole  $(0, \infty) \times \widehat{V' \times_F V''}$ . The first formula shows that on the boundary  $(0, \infty) \times S(F)$  the equation  $\hat{\alpha}(\rho, w', w'') = 0$  has a unique positive solution  $\rho = 1$  and the second formula shows that  $\frac{\partial}{\partial \rho} \hat{\alpha}(1, w', w'') \neq 0$  for any  $(w', w'') \in S(F)$ . Using the implicit function theorem we see that the equation  $\hat{\alpha}(\rho, w', w'') = 0$  defines a smooth positive function on a neighborhood of the boundary  $S(F)$  in  $\widehat{V' \times_F V''}$ .  $\blacksquare$

**Lemma 2.7.** *Let  $F \rightarrow X$  be a real vector bundle of finite rank  $m$ ,  $W \subset F$  an open neighborhood of the zero section of  $F$ , and  $\hat{W}$  the spherical blowup of  $W$  along the zero section  $X \subset F$ . Let  $\tau : W \rightarrow \mathbb{R}$  be a smooth map whose fiber restrictions  $\tau_x := \tau|_{W_x}$  satisfy the inequality*

$$(11) \quad |\tau_x(v)| \leq C_x \|v\|^k \quad \forall x \in X \quad \forall v \in W_x$$

*for a continuous function  $C : X \rightarrow (0, \infty)$  and a positive integer  $k$ . Then the map  $W \setminus X = \text{Int}(\hat{W}) \rightarrow \mathbb{R}$  given by*

$$v \mapsto \frac{1}{\|v\|^k} \tau(v)$$

*extends to a smooth map  $\hat{\tau} : \hat{W} \rightarrow \mathbb{R}$  whose restriction to the boundary  $\partial\hat{W} = S(F)$  is given by*

$$(12) \quad \hat{\tau}(w) = \lim_{r \rightarrow 0} \frac{1}{r^k} \tau(rw) = \frac{1}{k!} D_{0_x}^k \tau_x(\underbrace{w, w \dots w}_k) \quad \forall w \in S(F_x) .$$

*Proof.* Choose  $N > k$ . The order  $N$ -Taylor expansion at  $0_x$  of the fiber restriction  $\tau_x$  reads

$$\tau_x(v) = \sum_{0 \leq l \leq N} \frac{1}{l!} D_{0_x}^l \tau_x(\underbrace{v \dots v}_l) + \frac{1}{N!} \int_0^1 (1-t)^N D_{tv}^{N+1} \tau_x(\underbrace{v \dots v}_{N+1}) dt .$$

The assumption (11) implies

$$D_{0_x}^l \tau_x = 0 \quad \forall x \in X \quad \forall l \in \{0, \dots, k-1\} .$$

Putting  $v = rw$  with  $\|w\| = 1$  we get for  $r > 0$

$$\frac{1}{\|v\|^k} \tau(v) = \sum_{k \leq l \leq N} \frac{1}{l!} D_{0_x}^l \tau_x(\underbrace{w \dots w}_l) r^{l-k} + \frac{r^{N-k}}{N!} \int_0^1 (1-t)^N D_{trw}^{N+1} \tau_x(\underbrace{w \dots w}_{N+1}) dt ,$$

which can obviously be smoothly extended to a smooth function  $\hat{\tau}$  on spherical blow up  $\hat{W}$  whose restriction to the boundary  $\partial\hat{W} = S(F)$  is given by (12).  $\blacksquare$

**Remark 2.8.** *Proposition 2.5 shows that, “around its boundary”, the blowup flip passage  $\hat{Q}_f$  depends only on*

- (1) *the trivialization of the normal line bundle  $N_T$  induced by  $df$ ,*
- (2) *the restrictions  $\mathfrak{h}'$ ,  $\mathfrak{h}''$  of the metrics  $h'$ ,  $h''$  on  $F' := E'|_T$ ,  $F'' := E''|_T$ .*

### 3. THE HOLOMORPHIC MODEL THEOREM

**3.1. The blowup flip passage associated with a circle of regular reductions.** Let  $(X, g)$  be a Gauduchon surface with  $p_g(X) = 0$  and  $b_1(X) = 1$ , and let  $D$ ,  $L$  be Hermitian line bundles on  $X$ . We fix Hermite-Einstein connections  $a \in \mathcal{A}(D)$ ,  $b_0 \in \mathcal{A}(L)$  such that

$$\int_X i\Lambda_g F_{b_0} = \frac{1}{2} \int_X i\Lambda_g F_a ,$$

and we denote by  $\delta := \bar{\partial}_a$ ,  $\sigma_0 := \bar{\partial}_{b_0}$  the corresponding integrable semiconnections.

Put  $c := c_1(L)$ . In section 4.3 we identified the component  $\text{Pic}^c(X)$  of the Picard group of  $X$  with the moduli space  $\mathcal{M}(L)$ , which has a very simple description as a finite dimensional quotient:

$$\mathcal{M}(L) = \Sigma / G_{x_0} ,$$

for a base point  $x_0 \in X$ . This identification is given explicitly by  $[\sigma] \mapsto [\mathcal{L}_\sigma]$ . In our case  $\Sigma$  is an affine complex line, and  $G_{x_0}$  is a cyclic group canonically isomorphic to  $2\pi H^1(X, i\mathbb{Z})$ .

For  $\sigma \in \Sigma$  we denote by  $b_\sigma$  the Chern connection of the Hermitian line bundle  $\mathcal{L}_\sigma$ , which will also be Hermite-Einstein (see Remark 4.6 in the Appendix). Put

$$\check{L} := L^\vee \otimes D , \quad L' := \check{L}^\vee \otimes L , \quad L'' := L^\vee \otimes \check{L} .$$

Similarly, for a connection  $b$ , a semiconnection  $\sigma$  and a holomorphic structure  $\mathcal{L}$  on  $L$  we put

$$\check{b} := b^\vee \otimes a , \quad b' := \check{b}^\vee \otimes b , \quad b'' := b^\vee \otimes \check{b} , \quad \check{\sigma} := \sigma^\vee \otimes \delta , \quad \sigma' := \check{\sigma}^\vee \otimes \sigma , \quad \sigma'' := \sigma^\vee \otimes \check{\sigma} ,$$

$$\check{\mathcal{L}} := \mathcal{L}^\vee \otimes \mathcal{D} , \quad \mathcal{L}' := \check{\mathcal{L}}^\vee \otimes \mathcal{L} , \quad \mathcal{L}'' := \mathcal{L}^\vee \otimes \check{\mathcal{L}} .$$

Consider the circle

$$T := \left\{ [\mathcal{L}] \in \text{Pic}^c(X) \mid \deg_g(\mathcal{L}) = \frac{1}{2} \deg_g(\mathcal{D}) \right\} .$$

Our identification  $\mathcal{M}(L) = \text{Pic}^c(X)$  restricts to an identification

$$\Sigma_0 / G_{x_0} \xrightarrow{\cong} T ,$$

where  $\Sigma_0 := \sigma_0 + H_{\text{cl}}^{0,1}$ . For  $\sigma \in \Sigma_0$  put

$$v'_\sigma := \Lambda_g \partial_{b'_\sigma} : A^{0,1}(L') \rightarrow A^0(L') , \quad v''_\sigma := \Lambda_g \partial_{b''_\sigma} : A^{0,1}(L'') \rightarrow A^0(L'')$$

$$\mathfrak{H}'_\sigma := \ker(\sigma' : A^{0,1}(L') \rightarrow A^{0,2}(L')) \cap \ker(v'_\sigma) ,$$

$$\mathfrak{H}''_\sigma := \ker(\sigma'' : A^{0,1}(L'') \rightarrow A^{0,2}(L'')) \cap \ker(v''_\sigma) .$$

Using Proposition 4.2 proved in the Appendix we obtain

**Lemma 3.1.** *Suppose  $2c \neq c_1(D)$  and let  $\sigma \in \sigma_0 + H_{\text{cl}}^{0,1}$ . Then*

1.  $h^0(\mathcal{L}'_\sigma) = h^0(\mathcal{L}''_\sigma) = 0$ ,
2. *The natural morphisms  $\mathfrak{H}'_\sigma \rightarrow H^1(\mathcal{L}'_\sigma)$ ,  $\mathfrak{H}''_\sigma \rightarrow H^1(\mathcal{L}''_\sigma)$  are isomorphisms.*

*Proof.* Since  $\deg_g(\mathcal{L}'_\sigma) = \deg_g(\mathcal{L}''_\sigma) = 0$ , the Einstein constants of  $b'_\sigma$ ,  $b''_\sigma$  vanish. Therefore Proposition 4.2 applies, and shows that any holomorphic section of  $\mathcal{L}'_\sigma$  ( $\mathcal{L}''_\sigma$ ) is  $b'_\sigma$ -parallel ( $b''_\sigma$ -parallel). But a non-trivial parallel section of  $\mathcal{L}'_\sigma$  ( $\mathcal{L}''_\sigma$ ) would define a bundle isomorphism  $\mathcal{L}^{\otimes 2} \rightarrow \mathcal{D}$ , which contradicts the hypothesis. The second statement follows directly from Proposition 4.2.  $\blacksquare$

Let now  $\mathcal{L} = \mathcal{L}_{x_0}$  the Poincaré line bundle (associated with the base point  $x_0$ ) on  $\mathcal{M}(L) \times X$  and denote by  $p_1 : \mathcal{M}(L) \times X \rightarrow \mathcal{M}(L)$ ,  $p_2 : \mathcal{M}(L) \times X \rightarrow X$  the two projections. Consider the coherent sheaves

$$\mathcal{H}' := R^1(p_1)^*(\mathcal{L}^{\otimes 2} \otimes p_2^*(\mathcal{D}^\vee)) , \quad \mathcal{H}'' := R^1(p_1)^*(\mathcal{L}^{\otimes -2} \otimes p_2^*(\mathcal{D}))$$

on  $\mathcal{M}(L)$ .

**Definition 3.2.** *A pair  $(L, \mathcal{D})$  as above with  $2c_1(L) \neq c_1(\mathcal{D})$  will be called regular if  $h^2(\mathcal{L}') = h^2(\mathcal{L}'') = 0$  for any  $[\mathcal{L}] \in T$ .*

The regularity of the pair  $(L, \mathcal{D})$  is equivalent to the condition “ $\mathcal{T}_\lambda$  is a circle of regular reductions” mentioned in the introduction (see Corollary 1.21 in [Te3]). Using Lemma 3.1, the Riemann-Roch theorem and Grauert’s semicontinuity, local triviality and base change theorems, we obtain

**Proposition 3.3.** *Let  $(L, \mathcal{D})$  be a regular pair. For any sufficiently small  $\varepsilon > 0$  the restrictions of  $\mathcal{H}'$ ,  $\mathcal{H}''$  to the annulus*

$$\mathcal{M}(L)_\varepsilon := \left\{ [\mathcal{L}] \in \text{Pic}^c(X) \mid \pi |\deg_g(\mathcal{L}) - \frac{1}{2} \deg_g(\mathcal{D})| < \varepsilon \right\}$$

*are locally free of ranks*

$$r' = -\frac{1}{2}(2c - c_1(E))(2c - c_1(E) + c_1(X)), \quad r'' = -\frac{1}{2}(-2c + c_1(E))(-2c + c_1(E) + c_1(X))$$

*respectively, and for any  $l \in \mathcal{M}(L)_\varepsilon$  one has canonical identifications*

$$\mathcal{H}'(l) = H^1(\mathcal{L}'_l), \quad \mathcal{H}''(l) = H^1(\mathcal{L}''_l).$$

**Remark 3.4.** *Since the annulus  $\mathcal{M}(L)_\varepsilon$  is Stein and homotopically equivalent to a circle, it follows that the bundles  $\mathcal{H}'|_{\mathcal{M}(L)_\varepsilon}$ ,  $\mathcal{H}''|_{\mathcal{M}(L)_\varepsilon}$  are in fact trivial [Gr].*

In section 4.3 we showed that the Poincaré line bundle  $\mathcal{L}$  comes with a canonical Hermitian metric and is fiberwise Hermite-Einstein in the  $X$ -directions. Therefore  $\mathcal{L}'_l$ ,  $\mathcal{L}''_l$  become Hermitian line bundles. Using the isomorphisms given by the second conclusion of Lemma 3.1 and the  $L^2$ -inner product on the spaces  $\mathfrak{H}'_\sigma$ ,  $\mathfrak{H}''_\sigma$  we get Hermitian metrics  $\mathfrak{h}'$ ,  $\mathfrak{h}''$  on the bundles  $\mathcal{H}'|_T$ ,  $\mathcal{H}''|_T$ .

Define  $f_{\mathcal{D}} : \mathcal{M}(L) \rightarrow \mathbb{R}$  by  $f_{\mathcal{D}}([\sigma]) := 2\pi(\deg(\mathcal{L}_\sigma) - \frac{1}{2}\deg(\mathcal{D}))$ . According to Remark 2.8 the system

$$(\mathcal{M}(L)_\varepsilon, \mathcal{H}'|_{\mathcal{M}(L)_\varepsilon}, \mathcal{H}''|_{\mathcal{M}(L)_\varepsilon}, \mathfrak{h}', \mathfrak{h}'', f_{\mathcal{D}}|_{\mathcal{M}(L)_\varepsilon})$$

can be used to define a blowup flip passage (around its boundary) in a coherent way. We will denote by  $\hat{Q}$  this blowup flip passage. Recall that its interior comes with a complex structure, and its boundary can be identified with the projective bundle  $\mathbb{P}(\mathcal{H}'|_T \oplus \overline{\mathcal{H}''|_T})$  over  $T$ .

**3.2. The moduli space  $\mathcal{N}$ .** The family of operators

$$(13) \quad A^0(L') \xrightarrow{\sigma'} A^{0,1}(L'), \quad A^0(L'') \xrightarrow{\sigma''} A^{0,1}(L'')$$

$$(14) \quad A^{0,1}(L') \xrightarrow{\sigma'} A^{0,2}(L'), \quad A^{0,1}(L'') \xrightarrow{\sigma''} A^{0,2}(L'')$$

associated with points  $\sigma \in \Sigma$  are  $G^\mathbb{C}$ -equivariant, if we let the group  $G^\mathbb{C}$  act on  $\Sigma$  by

$$\varphi \cdot \sigma := \varphi \circ \sigma \circ \varphi^{-1} = \sigma - \varphi^{-1} \bar{\partial} \varphi$$

and on the spaces  $A^{0,q}(L')$ ,  $A^{0,q}(L'')$  by

$$\varphi \cdot \alpha' = \varphi^2 \alpha', \quad \varphi \cdot \alpha'' = \varphi^{-2} \alpha'.$$

Let  $\Sigma_\varepsilon$  the preimage of the annulus  $\mathcal{M}(L)_\varepsilon$  under the quotient map  $\Sigma \rightarrow \mathcal{M}(L)$ . Therefore  $\Sigma_\varepsilon$  is a symmetric neighborhood of the real line  $\Sigma_0$  in  $\Sigma$ . Suppose now that the pair  $(L, \mathcal{D})$  is regular in the sense of Definition 3.2. By Lemma 3.1 and Grauert’s semicontinuity theorem it follows for sufficiently small  $\varepsilon > 0$  the following holds: for any  $\sigma \in \Sigma_\varepsilon$  the two operators in (13) are injective, and the two operators in (14) are surjective. From now on we suppose that  $\varepsilon$  is sufficiently small such that these properties hold on  $\Sigma_\varepsilon$ . We will need two *holomorphic*,  $G^\mathbb{C}$ -equivariant

families of operators  $v'_\sigma, v''_\sigma$  defined for every  $\sigma \in \Sigma_\varepsilon$  such that  $\ker(v'_\sigma), \ker(v''_\sigma)$  are complements of the images of the two operators in (13) for every  $\sigma \in \Sigma_\varepsilon$ . The families  $\sigma \mapsto \Lambda_g \partial_{b'_\sigma}, \sigma \mapsto \Lambda_g \partial_{b''_\sigma}$  are  $G$ -equivariant, but unfortunately they are not holomorphic. They are antiholomorphic. Since we are interested in holomorphic models, this complicates our arguments.

**Proposition 3.5.** *For sufficiently small  $\varepsilon > 0$  there exists  $G^\mathbb{C}$ -equivariant families of operators*

$$v'_\sigma : A^{0,1}(L') \rightarrow A^0(L') , \quad v''_\sigma : A^{0,1}(L'') \rightarrow A^0(L'') ,$$

$$w'_\sigma : A^{0,2}(L') \rightarrow A^{0,1}(L') , \quad w''_\sigma : A^{0,2}(L'') \rightarrow A^{0,1}(L'')$$

depending holomorphically on  $\sigma \in \Sigma_\varepsilon$  such that for any  $\sigma \in \Sigma_\varepsilon$  it holds

- (1)  $\ker(v'_\sigma), \ker(v''_\sigma)$  are topological complements of the images of the two operators of (13),
- (2)  $v'_\sigma \circ w'_\sigma = 0, v''_\sigma \circ w''_\sigma = 0,$
- (3)  $w'_\sigma, w''_\sigma$  are right inverses of the two operators in (14).

In particular,  $\text{im}(w'_\sigma), \text{im}(w''_\sigma)$  are topological complements of

$$\mathfrak{H}'_\sigma := \ker(\sigma' : A^{0,1}(L') \rightarrow A^{0,2}(L')) \cap \ker(v'_\sigma) ,$$

$$\mathfrak{H}''_\sigma := \ker(\sigma'' : A^{0,1}(L'') \rightarrow A^{0,2}(L'')) \cap \ker(v''_\sigma)$$

in  $\ker(v'_\sigma), \ker(v''_\sigma)$  respectively.

*Proof.* Taking suitable Sobolev completions, the maps

$$\Sigma_0 \ni \sigma \mapsto v'_\sigma := \Lambda_g \partial_{b'_\sigma} , \quad \Sigma_0 \ni \sigma \mapsto v''_\sigma := \Lambda_g \partial_{b''_\sigma}$$

become real analytic,  $G^\mathbb{C}$ -equivariant and take values in a complex Banach space of bounded operators. Therefore these maps extend holomorphically on  $\Sigma_\varepsilon$  for sufficiently small  $\varepsilon > 0$ , and, by the identity theorem for holomorphic applications, the extension will still be  $G^\mathbb{C}$ -equivariant. Since  $T = \Sigma_0/G_{x_0}$  is compact, condition (1) (which is open with respect to  $\sigma$ ) will hold on a sufficiently small  $\Sigma_\varepsilon$ . Similarly, for  $\sigma \in \Sigma_0$  let

$$w'_\sigma : A^{0,2}(L') \rightarrow A^{0,1}(L') , \quad w''_\sigma : A^{0,2}(L'') \rightarrow A^{0,1}(L'')$$

be the right inverses of the two operators in (14) which take values in the  $L^2$ -orthogonal complements of  $\mathfrak{H}'_\sigma$  in  $K'_\sigma := \ker(v'_\sigma)$ , and  $\mathfrak{H}''_\sigma$  in  $K''_\sigma := \ker(v''_\sigma)$  respectively. These maps are again real analytic on  $\Sigma_0$  hence, for sufficiently small  $\varepsilon > 0$ , they admit holomorphic extensions  $\Sigma_\varepsilon \ni \sigma \mapsto w'_\sigma, \Sigma_\varepsilon \ni \sigma \mapsto w''_\sigma$ . The equivariance properties and the claims (2), (3) follow using again the identity theorem for holomorphic maps.  $\blacksquare$

Note that, for  $\sigma \notin \Sigma_0$  we cannot expect the operators  $w'_\sigma, w''_\sigma$  to take values in the orthogonal complements of  $\mathfrak{H}'_\sigma, \mathfrak{H}''_\sigma$ .

Choose  $\varepsilon > 0$  for which the claims of Proposition 3.5 hold, and endow the product

$$\mathcal{C} := V^{0,1} \times \Sigma_\varepsilon \times A^{0,1}(L') \times A^{0,1}(L'')$$



(see Theorem 4.1 in the Appendix) with the natural  $G^{\mathbb{C}}$ -action which is trivial on the first factor, and acts as explained above on the other factors. The system

$$\begin{cases} v'_\sigma \alpha' &= 0 \\ v''_\sigma \alpha'' &= 0 \\ \bar{\partial}v + \alpha' \wedge \alpha'' &= 0 \\ \sigma' \alpha' + 2v \wedge \alpha' &= 0 \\ \sigma'' \alpha'' - 2v \wedge \alpha'' &= 0 \end{cases}, \quad (\mathfrak{N})$$

on  $\mathcal{C}$  is  $G^{\mathbb{C}}$ -equivariant. Our hypothesis  $p_g(X) = 0$ , implies that the restriction

$$\bar{\partial}_0 := \bar{\partial}|_{V^{0,1}} : V^{0,1} \rightarrow A^{0,2}(X)$$

is an isomorphism, hence the third equation of  $(\mathfrak{N})$  is equivalent to the quadratic equation  $v = -\bar{\partial}_0^{-1}(\alpha' \wedge \alpha'')$ . The space of solutions  $\mathcal{C}^{\mathfrak{N}}$  of  $(\mathfrak{N})$  is a finite dimensional complex space. A solution with trivial  $\alpha'$ ,  $\alpha''$ -components also has trivial  $v$ -component and, under our regularity assumption, the map  $\mathfrak{N}$  defined by the left hand terms of  $(\mathfrak{N})$  is submersive at any such solution. Therefore  $\mathcal{C}^{\mathfrak{N}}$  is smooth at such a point with tangent space

$$(15) \quad T_{(0,\sigma,0)}\mathcal{C}^{\mathfrak{N}} = H^{0,1} \oplus \mathfrak{H}'_\sigma \oplus \mathfrak{H}''_\sigma = H^{0,1} \oplus \mathfrak{H}_\sigma,$$

where we have put  $\mathfrak{H}_\sigma := \mathfrak{H}'_\sigma \oplus \mathfrak{H}''_\sigma$ . The quotient

$$\mathcal{N} := \mathcal{C}^{\mathfrak{N}} / G_{x_0}.$$

by the cyclic group  $G_{x_0}$  comes with a residual  $\mathbb{C}^*$ -action given explicitly by  $\zeta \cdot (v, \sigma, \alpha', \alpha'') := (v, \sigma, \zeta^2 \alpha', \zeta^{-2} \alpha'')$ . The fixed point locus  $\mathcal{N}^{\mathbb{C}^*}$  is the space of orbits of points with vanishing  $(\alpha', \alpha'')$ -component, hence it can be identified with  $\Sigma_\varepsilon / G^{\mathbb{C}} = \mathcal{M}(L)_\varepsilon$ , and  $\mathcal{N}$  is smooth around the fixed point locus.

The assignments  $\sigma \mapsto \mathfrak{H}'_\sigma$ ,  $\sigma \mapsto \mathfrak{H}''_\sigma$  are holomorphic and  $G_{x_0}$ -equivariant, and the canonical maps  $\mathfrak{H}'_\sigma \rightarrow H^1(\mathcal{L}'_\sigma)$ ,  $\mathfrak{H}''_\sigma \rightarrow H^1(\mathcal{L}''_\sigma)$  are isomorphisms. Factorizing by  $G_{x_0}$  we obtain holomorphic bundles  $\mathfrak{H}'$ ,  $\mathfrak{H}''$  on  $\mathcal{M}(L)_\varepsilon$  with obvious isomorphisms  $\mathfrak{H}' = \mathcal{H}'|_{\mathcal{M}(L)_\varepsilon}$ ,  $\mathfrak{H}'' = \mathcal{H}''|_{\mathcal{M}(L)_\varepsilon}$ . Put  $\mathfrak{H} := \mathfrak{H}' \oplus \mathfrak{H}''$ . Via the identification  $\mathcal{N}^{\mathbb{C}^*} = \mathcal{M}(L)_\varepsilon$ , the restriction of the holomorphic tangent bundle  $\mathcal{T}_{\mathcal{N}}$  to  $\mathcal{N}^{\mathbb{C}^*}$  is

$$(16) \quad \mathcal{T}_{\mathcal{N}}|_{\mathcal{N}^{\mathbb{C}^*}} = \mathcal{T}_{\mathcal{M}(L)} \oplus \mathfrak{H} = \mathcal{T}_{\mathfrak{H}}|_{\mathcal{M}(L)_\varepsilon}.$$

The following remark (whose proof will be omitted) explains the role of the moduli space  $\mathcal{N}$  in our arguments: its  $\mathbb{C}^*$ -quotient is mapped naturally to the quotient  $\mathcal{A}_\delta^{0,1}(E)^{\text{int}}$ .

**Remark 3.6.** Put  $E = L \oplus \check{L}$ . The map  $\eta : \mathcal{C} \rightarrow \mathcal{A}_\delta^{0,1}(E)$  given by

$$\eta(v, \sigma, \alpha', \alpha'') := \begin{pmatrix} \sigma + v & \alpha' \\ \alpha'' & \check{\sigma} - v \end{pmatrix}.$$

has the properties:

1. is holomorphic and equivariant with respect to the group monomorphism

$$\iota : G^{\mathbb{C}} \rightarrow \mathcal{G}_E^{\mathbb{C}} := \Gamma(X, \text{SL}(E)), \quad \iota(\varphi) := \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix},$$

2. maps  $\mathcal{C}^{\mathfrak{N}}$  into  $\mathcal{A}_\delta^{0,1}(E)^{\text{int}}$ , hence it induces a map

$$[\eta] : \mathcal{N} / \mathbb{C}^* \rightarrow \mathcal{A}_\delta^{0,1}(E)^{\text{int}} / \mathcal{G}_E^{\mathbb{C}},$$

3. *There exists an open neighborhood  $W$  of  $T$  in  $\mathcal{N}/\mathbb{C}^*$  such that the restriction*

$$[\eta^*] := [\eta]|_{W \setminus \mathcal{N}^{\mathbb{C}^*}}$$

*of  $[\eta]$  takes values in  $\mathcal{M}_{\mathcal{D}}^{\text{si}}(E)$  and is holomorphic.*

One can prove that, if  $W$  is sufficiently small, the restriction  $[\eta^*]$  is an open embedding into  $\mathcal{M}_{\mathcal{D}}^{\text{si}}(E)$  (hence in particular injective). Since we are interested only in moduli space polystable structures, we will not need this result. Note that  $W \setminus \mathcal{N}^{\mathbb{C}^*}$  is not Hausdorff in general.

$\mathcal{N}$  comes with a natural holomorphic map

$$\pi : \mathcal{N} \rightarrow \mathcal{M}(L)_{\varepsilon}, \quad [v, \sigma, \alpha', \alpha''] \mapsto [\sigma] .$$

More precisely  $\pi$  is a holomorphic (non-linear) subfibration of the vector bundle  $V^{0,1} \times (A' \times_{\mathcal{M}(L)_{\varepsilon}} A'')$  over  $\mathcal{M}(L)_{\varepsilon}$ , where  $A', A''$  are the bundles associated with the principal  $G_{x_0}$ -bundle  $p_{\varepsilon} : \Sigma_{\varepsilon} \rightarrow \mathcal{M}(L)_{\varepsilon}$  and the natural representations of  $G_{x_0}$  in  $A^{0,1}(L')$ ,  $A^{0,1}(L'')$  respectively. After suitable Sobolev completions  $A', A''$  become holomorphic Banach bundles, and  $\mathcal{N}$  is tangent to the finite rank subbundle  $\{0\} \times \mathfrak{H}$  along the zero section. Is important to note that, using the operator families  $(w'_{\sigma}), (w''_{\sigma})$ , the fibration  $\pi$  can be *holomorphically* “linearized” over  $\mathcal{M}(L)_{\varepsilon}$  in an explicit way:

**Remark 3.7.** *The map*

$$\mathcal{C} \rightarrow \Sigma_{\varepsilon} \times A^{0,1}(L') \oplus A^{0,1}(L'') , \quad (v, \sigma, \alpha', \alpha'') \mapsto \begin{pmatrix} \sigma \\ \alpha' + 2w'_{\sigma}(v \wedge \alpha') \\ \alpha'' - 2w''_{\sigma}(v \wedge \alpha'') \end{pmatrix}$$

*is  $G^{\mathbb{C}}$ -equivariant, induces a holomorphic map  $\mathfrak{A} : \mathcal{C}^{\mathfrak{N}} \rightarrow p_{\varepsilon}^*(\mathfrak{H})$  over  $\Sigma_{\varepsilon}$  with the properties:*

1. *For any  $\sigma \in \Sigma_{\varepsilon}$  one has  $D_{(0,\sigma,0)}\mathfrak{A} = \text{id}$ , in particular  $\mathfrak{A}$  is a local biholomorphism at any point of  $\mathcal{C}^{\mathbb{C}^*}$ .*
2.  *$\mathfrak{A}$  induces  $\mathbb{C}^*$ -equivariant holomorphic map  $\mathfrak{a} : \mathcal{N} \rightarrow \mathfrak{H}$  over  $\mathcal{M}(L)$ , which is a local biholomorphism at any point of  $\mathcal{N}^{\mathbb{C}^*}$ .*
3. *There exists an open,  $S^1$ -invariant neighborhood  $\mathcal{W}$  of  $T$  in  $\mathcal{N}$  such that*

$$\mathfrak{a}_{\mathcal{W}} := \mathfrak{a}|_{\mathcal{W}} : \mathcal{W} \rightarrow \mathfrak{a}(\mathcal{W})$$

*is a biholomorphism over  $\mathcal{M}(L)$ .*

4.  *$\mathfrak{a}_{\mathcal{W}}$  and its inverse  $\mathfrak{b}_{\mathcal{W}} := \mathfrak{a}_{\mathcal{W}}^{-1}$  admit  $G^{\mathbb{C}}$ -invariant lifts  $\mathfrak{A}_{\mathcal{W}}, \mathfrak{B}_{\mathcal{W}}$  to the pre-images of their domains in  $\mathcal{C}^{\mathfrak{N}}, p_{\varepsilon}^*(\mathfrak{H})$  respectively.*

Denoting by  $\mathfrak{H}^*$  the complement of the zero section in  $\mathfrak{H}$  note that

**Remark 3.8.** *The holomorphic map  $\{\mathfrak{a}^*\} : \{\mathcal{W} \setminus \mathcal{N}^{\mathbb{C}^*}\}/\mathbb{C}^* \rightarrow \mathfrak{H}^*/\mathbb{C}^*$  induced by  $\mathfrak{a}$  is étale.*

Note that we cannot expect this map to be injective.

**3.3. The moduli spaces  $\mathcal{N}^{\mathfrak{S}\mathfrak{J}}, \mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}$ .** To complete the proof of the holomorphic model theorem we need a finite dimensional description of the moduli spaces  $\mathcal{M}_a^{\text{ASD}}(E), \hat{\mathcal{M}}_a^{\text{ASD}}(E)_{\lambda}$  around  $\mathcal{T}_{\lambda}$  (respectively  $\mathcal{P}_{\lambda}$ ). On the product  $\mathcal{A}(L) \times A^1(L')$  consider the equations

$$\begin{cases} \Lambda_g(\partial_{b'}\beta^{01} - \bar{\partial}_{b'}\beta^{10}) & = 0 \\ \Lambda_g(\partial_{b'}\beta^{01} + \bar{\partial}_{b'}\beta^{10}) & = 0 \\ p_r\{\Lambda_g(F_b - \frac{1}{2}F_a) - i(|\beta^{01}|^2 - |\beta^{10}|^2)\} & = 0 \\ p_r\Lambda_g d^c(b - b_0) & = 0 \end{cases} \quad (\mathfrak{S})$$

$$\begin{cases} \bar{\partial}_{b'}\beta^{01} & = 0 \\ \partial_{b'}\beta^{10} & = 0 \\ F_b^{0,2} - \beta^{01} \wedge \beta^{*01} & = 0 \end{cases} \quad (\mathfrak{J})$$

$$\frac{1}{2} \int_X \{i\Lambda_g(F_b - \frac{1}{2}F_a) + (|\beta'|^2 - |\beta''|^2)\} \text{vol}_g = 0. \quad (\mathfrak{M})$$

We denote by

$$\mathfrak{S} : \mathcal{A}(L) \times A^1(L') \rightarrow A^0(L') \oplus A^0(L') \oplus A^0(X, i\mathbb{R})_r \oplus A^0(X, i\mathbb{R})_r,$$

$$\mathfrak{J} : \mathcal{A}(L) \times A^1(L') \rightarrow A^{0,2}(L') \oplus A^{0,2}(L'') \oplus A^{0,2}(X), \quad \mathfrak{M} : \mathcal{A}(L) \times A^1(L') \rightarrow \mathbb{R}$$

the maps defined by the left hand of  $(\mathfrak{S})$ ,  $(\mathfrak{J})$ ,  $(\mathfrak{M})$  respectively. Note that  $\mathfrak{S}$ ,  $\mathfrak{J}$  are  $G$ -equivariant and  $\mathfrak{M}$  is  $G$ -invariant. We denote by  $\{\mathcal{A}(L) \times A^1(L')\}^{\mathfrak{S}}$ ,  $\{\mathcal{A}(L) \times A^1(L')\}^{\mathfrak{J}}$ ,  $\{\mathcal{A}(L) \times A^1(L')\}^{\mathfrak{M}}$  the spaces of solutions of the system indicated as exponent, and by  $\mathcal{N}^{\mathfrak{S}\mathfrak{J}}$ ,  $\mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}$  the  $G_{x_0}$ -quotients of the latter two spaces (which are finite dimensional). One has obvious identifications

$$\{\mathcal{N}^{\mathfrak{S}\mathfrak{J}}\}^{S^1} = \mathcal{M}^{\text{HE}}(L), \quad \{\mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}\}^{S^1} = T,$$

where

$$T := b_0 + H_{\text{cl}}/G = \left\{ [b] \in \mathcal{M}^{\text{HE}}(L) \mid \int_X (i\Lambda_g F_b) \text{vol}_g = \frac{1}{2} \int_X (i\Lambda_g F_a) \text{vol}_g \right\},$$

corresponds via the Kobayashi-Hitchin identification  $\mathcal{M}^{\text{HE}}(L) = \mathcal{M}(L)$  to the circle denoted in the previous section by the same symbol. Note that (under our regularity condition)  $\mathcal{N}^{\mathfrak{S}\mathfrak{J}}$  and  $\mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}$  are smooth at any point of  $T$ . The restriction to  $T$  of the corresponding tangent bundles are:

$$(17) \quad T_{\mathcal{N}^{\mathfrak{S}\mathfrak{J}}} |_T = T_{\mathcal{M}^{\text{HE}}(L)} \oplus (\tilde{\mathfrak{H}}_T'' \oplus \mathfrak{H}_T'), \quad T_{\mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}} |_T = T_T \oplus (\tilde{\mathfrak{H}}_T'' \oplus \mathfrak{H}_T'),$$

where  $\mathfrak{H}_T'$ ,  $\mathfrak{H}_T''$  are the restriction to  $T$  of the bundles  $\mathfrak{H}'$ ,  $\mathfrak{H}''$  defined in section 3.2, and are obtained from the families of vector spaces

$$b_0 + H_{\text{cl}}^1 \ni b \rightarrow \mathfrak{H}_b' := \ker \bar{\partial}_{b'} \cap \ker(\Lambda_g \partial_{b'}), \quad b_0 + H_{\text{cl}}^1 \ni b \rightarrow \mathfrak{H}_b'' := \bar{\partial}_{b''} \cap \ker(\Lambda_g \partial_{b''}).$$

The bundle  $\tilde{\mathfrak{H}}_T''$  is obtained using the vector spaces  $\tilde{\mathfrak{H}}_b'' := \{(\beta'')^* \mid \beta \in \mathfrak{H}_b''\}$ . Put

$$\begin{aligned} \{\mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}\}^* &:= \mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}} \setminus \{\mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}\}^{S^1}, \quad \mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{J}\mathfrak{M}} := \{[b, \beta] \in \mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}} \mid \|\beta\|_{L^\infty} < \epsilon\}, \\ \{\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}\}^* &:= \mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{J}\mathfrak{M}} \setminus \{\mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}\}^{S^1}. \end{aligned}$$

**Proposition 3.9.** *Put  $E := L \oplus \tilde{L}$ . The map  $\theta : \mathcal{A}(L) \times A^1(L') \rightarrow \mathcal{A}_a(E)$  given by*

$$(b, \beta) \mapsto \begin{pmatrix} d_b & \beta \\ -\beta^* & d_{\tilde{b}} \end{pmatrix}$$

*has the following properties:*

(1) *is an affine isomorphism, which is equivariant with respect to the group*

$$\text{morphism } G \rightarrow \mathcal{G}_E \text{ given by } \varphi \mapsto \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix},$$

- (2) maps  $\mathcal{A}(L) \times A^1(L')^{\mathfrak{S}\mathfrak{M}}$  into  $\mathcal{A}_a^{\text{ASD}}(E)$  and, for sufficiently small  $\epsilon > 0$ , the induced map

$$[\theta] : \mathcal{N}^{\mathfrak{S}\mathfrak{M}}/S^1 \rightarrow \mathcal{M}_a^{\text{ASD}}(E)$$

maps isomorphically  $\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{M}}/S^1$  onto an open neighborhood  $\mathcal{O}_\lambda$  of  $\mathcal{T}_\lambda$  in  $\mathcal{M}_a^{\text{ASD}}(E)$ , and restricts to a diffeomorphism

$$[\theta^*] : \{\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{M}}\}^*/S^1 \rightarrow (\mathcal{O}_\lambda \setminus \mathcal{T}_\lambda) \subset \mathcal{M}_a^{\text{ASD}}(E)^* .$$

We will omit the proof of this statement. We mention only that the pull back of the ASD equation via  $\theta$  obviously coincides with the system formed by  $(\mathfrak{J})$ ,  $(\mathfrak{M})$ , the second and the third equation of  $(\mathfrak{S})$ . The role of the first and fourth equation of  $(\mathfrak{S})$  is to reduce (around  $\{b_0 + H_{\text{cl}}\} \times \{0\}$ ) the  $\mathcal{G}_E$ -factorization involved in the definition of  $\mathcal{M}_a^{\text{ASD}}(E)$  to the  $G$ -factorization.  $\blacksquare$

Taking into account this result we can *define* the blowup moduli space  $\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  using the blowup  $S^1$ -quotient of  $\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{M}}$ . More precisely, we put

$$\hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda := \{\mathcal{M}_a^{\text{ASD}}(E) \setminus \mathcal{O}_\lambda\} \amalg_{[\theta^*]} \widehat{\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{M}}/S^1} .$$

One can prove that this construction is equivalent to the one given in [Te3] section 1.4.2. With this definition we have obviously

**Remark 3.10.** *The restriction  $[\theta^*] : \{\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{M}}\}^*/S^1 \rightarrow \mathcal{M}_a^{\text{ASD}}(E)^*$  of  $[\theta]$  extends continuously to a smooth open embedding  $\widehat{[\theta]} : \widehat{\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{M}}/S^1} \rightarrow \hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  which identifies  $\partial\widehat{\mathcal{N}_\epsilon^{\mathfrak{S}\mathfrak{M}}}$  with the boundary  $\mathcal{P}_\lambda$  of  $\mathcal{M}_a^{\text{ASD}}(E)_\lambda$ .*

#### 3.4. The construction of the isomorphism. Put

$$\mathcal{G}_E^0 := \left\{ f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathcal{G}_E^{\mathbb{C}} \mid \int_X (f_{11} - 1) = 0 \right\}$$

$\mathcal{G}_E^0$  is not a subgroup of  $\mathcal{G}_E^{\mathbb{C}}$ ; it is a complex hypersurface which is transversal to the subgroup  $\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^* \right\} \simeq \mathbb{C}^*$  at  $\text{id}_E$ , and is invariant under the inner action of the subgroup

$$(18) \quad \left\{ \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix} \mid \phi \in \mathcal{C}^\infty(X, \mathbb{C}^*) \right\} \simeq \mathcal{G}^{\mathbb{C}} .$$

Let

$$S : \mathcal{A}(L) \times A^1(L') \times \mathcal{G}_E^0 \rightarrow A^0(L') \oplus A^0(L') \oplus A^0(X, i\mathbb{R})_r \oplus A^0(X, i\mathbb{R})_r$$

be the map defined by

$$S(b, \beta, f) := \mathfrak{S}(f \cdot (b, \beta)) ,$$

where on the right we used the  $\mathcal{G}_E^{\mathbb{C}}$ -action on  $\mathcal{A}(L) \times A^1(L)$  induced via  $\theta$  and (18) from the standard action of  $\mathcal{G}_E^{\mathbb{C}}$  on  $\mathcal{A}_a(E)$  (see [D]). We recall that the latter action is induced from the standard  $\mathcal{G}_E^{\mathbb{C}}$ -action on  $\mathcal{A}_\delta^{0,1}(E)$  (see section 4.2) via the real isomorphism of affine spaces

$$j : \mathcal{A}_a(E) \rightarrow \mathcal{A}_\delta^{0,1}(E) , \quad A \mapsto \bar{\partial}_A .$$

**Proposition 3.11.** *The map  $S$  has the following properties*

1. Is  $G$ -equivariant with respect to the action

$$\varphi \cdot (\beta_1, \beta_2, u_1, u_2) = (\varphi^2 \beta_1, \varphi^2 \beta_2, u_1, u_2)$$

of  $G$  on  $A^0(L') \oplus A^0(L') \oplus A^0(X, i\mathbb{R})_r \oplus A^0(X, i\mathbb{R})_r$ .

2. For every  $b \in \mathcal{A}(L)$  one has

$$\frac{\partial S}{\partial f}(b, 0, \text{id}_E) = P_b := \delta_b d_b ,$$

where

$$d_b : T_{\text{id}_E} \mathcal{G}_E^0 = A^0(X, \mathbb{C})_r \times A^0(L') \times A^0(L'') \rightarrow A^1(X, i\mathbb{R}) \times A^1(L') ,$$

$$\delta_b : A^1(X, i\mathbb{R}) \times A^1(L') \rightarrow A^0(L') \times A^0(L') \times A^0(X, i\mathbb{R})_r \times A^0(X, i\mathbb{R})_r$$

are first order operators given by

$$d_b(f, v', v'') := \begin{pmatrix} -\partial \bar{f} + \bar{\partial} f \\ -\partial_{b'}(v'')^* + \bar{\partial}_{b'} v' \end{pmatrix}, \quad \delta_b(\dot{b}, \dot{\beta}) := \begin{pmatrix} \Lambda_g(\partial_{b'} \dot{\beta}^{01} - \bar{\partial}_{b'} \dot{\beta}^{10}) \\ \Lambda_g(\partial_{b'} \dot{\beta}^{01} + \bar{\partial}_{b'} \dot{\beta}^{10}) \\ p_r \Lambda_g d\dot{b} \\ p_r \Lambda_g d^c \dot{b} \end{pmatrix}.$$

This composition is an elliptic second order operator, and is an isomorphism when  $b \in b_0 + H_{\text{cl}}$ .

3. There exists an open,  $G$ -invariant neighborhood  $\mathcal{U}$  of  $\{b_0 + H_{\text{cl}}\} \times \{0\}$  in  $\mathcal{A}(L) \times A^1(L')$ , and an open,  $G$ -invariant neighborhood  $\mathcal{V}$  of  $\text{id}_E$  in  $\mathcal{G}_E^0$  such that the intersection of the zero locus  $Z(S)$  with  $\mathcal{U} \times \mathcal{V}$  is the graph of a smooth,  $G$ -equivariant function  $r : \mathcal{U} \rightarrow \mathcal{V}$  satisfying

$$r|_{\mathcal{U} \cap (b_0 + H_{\text{cl}})} \equiv \text{id}_E.$$

The first two claims can be checked by direct computations. The third claim follows from the first and the second. The map  $r$  is obtained locally, around the points of  $\{b_0 + H_{\text{cl}}\} \times \{0\}$ , by applying the implicit function theorem to  $S$ .  $\blacksquare$

Using  $r$  we obtain a  $G$ -equivariant map  $R : \mathcal{U} \rightarrow (\mathcal{A}(L) \times A^1(L'))^{\mathfrak{S}}$  given by

$$R(b, \beta) := r(b, \beta) \cdot (b, \beta) .$$

By construction, this map has the remarkable property

**Remark 3.12.** For any  $(b, \beta) \in \mathcal{U}$  the connections  $j(\theta(b, \beta))$ ,  $j(\theta(R(b, \beta))) \in \mathcal{A}_\delta^{0,1}(E)$  belong to the same  $\mathcal{G}_E^{\mathfrak{C}}$ -orbit.

Using Proposition 3.11 (2) we see that, for a point  $b \in b_0 + H_{\text{cl}}$ , one has

$$(19) \quad D_{(b,0)} R = p_{\ker(\delta_b)} ,$$

where  $p_{\ker(\delta_b)}$  stands for the projection  $A^1(X, i\mathbb{R}) \times A^0(L') \rightarrow \ker(\delta_b)$  associated with the direct sum decomposition  $A^1(X, i\mathbb{R}) \times A^1(L') = \text{im}(d_b) \oplus \ker(\delta_b)$ .

We will compare now the moduli spaces  $\mathcal{N}$ ,  $\mathcal{N}^{\mathfrak{S}\mathfrak{J}}$  using the  $\mathbb{R}$ -affine,  $G$ -equivariant embedding

$$\iota : \mathcal{C} \rightarrow \mathcal{A}(L) \times A^0(L'), \quad \iota(v, \sigma, \alpha', \alpha'') := \begin{pmatrix} (\sigma + \partial_\sigma) + (-\bar{v} + v) \\ \alpha' - (\alpha'')^* \end{pmatrix} ,$$

where  $\partial_\sigma$  denotes the unique operator  $A^0(L) \rightarrow A^{1,0}(L)$  for which  $\sigma + \partial_\sigma$  is a Hermitian linear connection on  $L$ . The pull-back of the “integrability” system  $(\mathfrak{J})$  via  $\iota$  is precisely the system  $(\mathfrak{N})$  involved in the definition of  $\mathcal{N}$ , therefore  $\iota$  induces

a map  $\mathcal{C}^{\mathfrak{M}} \rightarrow (\mathcal{A}(L) \times A^1(L'))^{\mathfrak{J}}$  which will be denoted by the same symbol  $\iota$ . Note now that, since  $(\mathfrak{J})$  is  $\mathcal{G}_E^{\mathbb{C}}$ -invariant, the composition

$$\mathfrak{R} := R \circ \iota|_{\iota^{-1}(\mathcal{U}) \cap \mathcal{C}^{\mathfrak{M}}} : \iota^{-1}(\mathcal{U}) \cap \mathcal{C}^{\mathfrak{M}} \rightarrow (\mathcal{A}(L) \times A^1(L'))^{\mathfrak{S}}$$

also takes in fact values in  $\mathcal{A}(L) \times A^1(L')^{\mathfrak{S}\mathfrak{J}}$ . Using (19) and the identification (15) we get

$$D_{(0,\sigma,0,0)}\mathfrak{R}(\dot{\sigma}, \dot{\alpha}', \dot{\alpha}'') = \begin{pmatrix} -\bar{\sigma} + \dot{\sigma} \\ -(\dot{\alpha}'')^* + \dot{\alpha}' \end{pmatrix} \quad \forall \sigma \in \sigma_0 + H_{\text{cl}}^{0,1},$$

for any  $\sigma \in \sigma_0 + H_{\text{cl}}^{0,1}$ ,  $\dot{\sigma} \in H^{0,1}$ ,  $\dot{\alpha}' \in \mathfrak{H}'_{\sigma}$ ,  $\dot{\alpha}'' \in \mathfrak{H}''_{\sigma}$ . This shows that the restriction of  $\mathfrak{R}$  to a sufficiently small open neighborhood of  $T$  (which can be supposed to be  $G$ -invariant) is a diffeomorphism. Taking  $G_{x_0}$ -quotients we obtain a smooth,  $S^1$ -equivariant map

$$\begin{array}{ccc} \iota^{-1}(\mathcal{U}) \cap \mathcal{C}^{\mathfrak{M}} / G_{x_0} & \xrightarrow{\mathfrak{r}} & \mathcal{N}^{\mathfrak{S}\mathfrak{J}} \\ \downarrow & & \\ \mathcal{N} & & \end{array}$$

induced by  $\mathfrak{R}$  and defined on an open,  $S^1$ -invariant neighborhood of  $T$  in  $\mathcal{N}$ . Let now

$$\mathcal{W} \subset \iota^{-1}(\mathcal{U}) \cap \mathcal{C}^{\mathfrak{M}} / G_{x_0}$$

be a sufficiently small, open,  $S^1$ -invariant neighborhood of  $T$  in  $\mathcal{N}$  such that

- The restriction  $\mathfrak{r}_{\mathcal{W}} := \mathfrak{r}|_{\mathcal{W}}$  is a diffeomorphism on its image  $\mathcal{N}_{\mathcal{W}}^{\mathfrak{S}\mathfrak{J}} := \mathfrak{r}(\mathcal{W})$ .
- $\mathcal{W} \cap \mathcal{N}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}} \subset \mathcal{N}_{\epsilon}^{\mathfrak{S}\mathfrak{J}\mathfrak{M}}$ , where  $\epsilon > 0$  satisfies the property stated in Proposition 3.9,
- $\mathcal{W}$  satisfies the property stated in Remark 3.7,
- the image  $W$  of  $\mathcal{W}$  in  $\mathcal{N}/\mathbb{C}^*$  satisfies the property stated in Remark 3.6.

Consider now the following smooth,  $S^1$ -invariant  $\mathbb{R}$ -valued maps

$$\mathfrak{m}_{\mathcal{W}} = \mathfrak{m}|_{\mathcal{N}_{\mathcal{W}}^{\mathfrak{S}\mathfrak{J}}} : \mathcal{N}_{\mathcal{W}}^{\mathfrak{S}\mathfrak{J}} \rightarrow \mathbb{R}, \quad \psi := \mathfrak{m}_{\mathcal{W}} \circ \mathfrak{r}_{\mathcal{W}} : \mathcal{W} \rightarrow \mathbb{R}, \quad \varphi := \mathfrak{m}_{\mathcal{W}} \circ \mathfrak{r}_{\mathcal{W}} \circ \mathfrak{b}_{\mathcal{W}} : \mathfrak{a}(\mathcal{W}) \rightarrow \mathbb{R}.$$

(see Remark 3.7 for the notations  $\mathfrak{a}_{\mathcal{W}}$ ,  $\mathfrak{b}_{\mathcal{W}}$ ,  $\mathfrak{z}_{\mathcal{W}}$ ,  $\mathfrak{B}_{\mathcal{W}}$ ). Using the functoriality of the spherical blowup with respect to diffeomorphisms [AK], we obtain diffeomorphisms of manifolds with boundary

$$\widehat{Z(\varphi)} \xrightarrow{\widehat{\mathfrak{b}_{\mathcal{W}}}} \widehat{Z(\psi)} \xrightarrow{\widehat{\mathfrak{r}_{\mathcal{W}}}} \widehat{Z(\mathfrak{m}_{\mathcal{W}})}, \quad \widehat{Z(\varphi)}/S^1 \xrightarrow{[\widehat{\mathfrak{b}_{\mathcal{W}}]}} \widehat{Z(\psi)}/S^1 \xrightarrow{[\widehat{\mathfrak{r}_{\mathcal{W}}]}} \widehat{Z(\mathfrak{m}_{\mathcal{W}})}/S^1,$$

extending the diffeomorphisms  $\mathfrak{b}_{\mathcal{W}}^*$ ,  $\mathfrak{r}_{\mathcal{W}}^*$ ,  $[\mathfrak{b}_{\mathcal{W}}^*]$ ,  $[\mathfrak{r}_{\mathcal{W}}^*]$  between the corresponding interiors. In particular, using Remark 3.10, we get an open embedding

$$(20) \quad [\widehat{\theta}] \circ [\widehat{\mathfrak{r}_{\mathcal{W}}}] \circ [\widehat{\mathfrak{b}_{\mathcal{W}}}] : \widehat{Z(\varphi)}/S^1 \hookrightarrow \hat{\mathcal{M}}_a^{\text{ASD}}(E)_{\lambda}.$$

*Proof.* (of Theorem 1.1) First of all note that rescaling the metric  $h$  on  $E$  if necessary we may suppose that the Chern connection  $a$  of the pair  $(\mathcal{D}, \det(h))$  is Hermite-Einstein, hence the formalism developed in sections 3.1, 3.2 applies.

Using Lemma 3.13 below and Proposition 2.5 it follows that (choosing a smaller  $\mathcal{W}$  if necessary) the blowup  $S^1$ -quotient  $\widehat{Z(\varphi)}/S^1$  is identified with a neighborhood of  $\partial\hat{Q}$  in the blowup flip passage  $\hat{Q}$  associated with this system. Therefore, using

(20), we get an open embedding  $\hat{Q} \supset O \xrightarrow{\chi} \mathcal{O}_\lambda \subset \hat{\mathcal{M}}_a^{\text{ASD}}(E)_\lambda$  satisfying the first claim of Theorem 1.1.

For the second claim of Theorem 1.1 we have to check that the composition

$$Z(\varphi)^*/S^1 \xrightarrow{[\theta^*] \circ [\mathfrak{r}_W^*] \circ [\mathfrak{b}_W^*]} \mathcal{M}_a^{\text{ASD}}(E)^*$$

becomes holomorphic if one endows  $Z(\varphi)^*/S^1$  with the holomorphic structure induced by the embedding  $Z(\varphi)^*/S^1 \hookrightarrow \mathfrak{H}^*/\mathbb{C}^*$ , and  $\mathcal{M}_a^{\text{ASD}}(E)^*$  with the holomorphic structure induced by the embedding  $[j^*] : \mathcal{M}_a^{\text{ASD}}(E)^* \hookrightarrow \mathcal{M}_{\mathcal{D}}^{\text{si}}(E)$ . The required holomorphy property follows by Remarks 3.6 (the holomorphy of  $[\eta^*]$ ), 3.8 (the holomorphy and étale property of  $\{\mathfrak{a}^*\}$ ), using the commutative diagram

$$\begin{array}{ccccc} Z(\varphi)^*/S^1 & \xleftarrow{[\mathfrak{a}_W^*] \simeq} & Z(\psi)^*/S^1 & \xrightarrow{[\mathfrak{r}_W^*] \simeq} & Z(\mathfrak{m}_W)^*/S^1 \xrightarrow{[\theta^*]} \mathcal{M}_a^{\text{ASD}}(E)^* \\ \downarrow & & \downarrow & & \downarrow [j^*] \\ \mathfrak{H}^*/\mathbb{C}^* & \xleftarrow{\{\mathfrak{a}^*\}} & W \setminus \mathcal{N}^{\mathbb{C}^*} & \xrightarrow{[\eta^*]} & \mathcal{M}_{\mathcal{D}}^{\text{si}}(E) . \end{array}$$

The key ingredient in the proof is the commutativity of the right hand rectangle, which follows from Remark 3.12:

$$\begin{aligned} [j^*]([\theta^*](\mathfrak{r}_W^*([v, \sigma, \alpha', \alpha'']))) &= [(j \circ \theta)(R(\iota(v, \sigma, \alpha', \alpha'')))] = [(j \circ \theta)(\iota(v, \sigma, \alpha', \alpha''))] \\ &= [\eta(v, \sigma, \alpha', \alpha'')] = [\eta^*]([v, \sigma, \alpha', \alpha'']) . \end{aligned}$$

■

**Lemma 3.13.** *The map  $\varphi$  satisfies the properties P1, P2, P3 of section 2.2 written for the system*

$$(\mathcal{M}(L)_\varepsilon, \mathcal{H}'|_{\mathcal{M}(L)_\varepsilon}, \mathcal{H}''|_{\mathcal{M}(L)_\varepsilon}, \mathfrak{h}', \mathfrak{h}'', f_{\mathcal{D}}|_{\mathcal{M}(L)_\varepsilon})$$

defined in section 3.1.

*Proof.* P1 and P2 follow immediately from the formula

$$(21) \quad (D_{(b,0)}\mathfrak{M})(\dot{b}, \dot{\beta}) = \frac{1}{2} \int i\Lambda_g(d\dot{b})\text{vol}_g \quad \forall b \in b_0 + H_{\text{cl}} \quad \forall \dot{b} \in A^1(X, i\mathbb{R}) \quad \forall \dot{\beta} \in \mathfrak{H}_b .$$

P3 is more delicate, because we have to compute the second derivative of a complicated composition of non-linear maps, including  $\mathfrak{r}_W$  which is induced by  $R$  (defined implicitly via  $r$ , not explicitly). The result follows from the following formulae, whose proofs will be omitted:

- (1) Let  $b \in b_0 + H_{\text{cl}}$  and  $\beta \in A^1(L')$ . Putting  $\xi = (0, \dot{\beta})$  we have

$$d_b P_b^{-1}((D_{(b,0)}^2 \mathfrak{S})(\xi, \xi)) = (id^c Q^{-1} p_r \{|\dot{\beta}^{01}|^2 - |\dot{\beta}^{10}|^2\}, 0) ,$$

where  $Q := p_r \Lambda_g d^c : A^0(X, \mathbb{R})_r \xrightarrow{\simeq} A^0(X, \mathbb{R})_r$  (see section 4.1).

- (2) Let  $b \in b_0 + H_{\text{cl}}$  and let  $(x_t)_{t \in (-\varepsilon, \varepsilon)}$  be a smooth path in  $\mathcal{A}(L) \times A^1(L')$  such that  $x_0 = (b, 0)$ ,  $\dot{x}_0 \in \ker(\delta_b)$ . Then

$$(a) \quad \left. \frac{d^2}{dt^2} \right|_{t=0} R(x(t)) = p_{\ker(\delta_b)}(\ddot{x}_0) - d_b(P_b^{-1}((D_{x_0}^2 \mathfrak{S})(\dot{x}_0, \dot{x}_0))) .$$

(b) If  $\dot{x}_0 = (0, \dot{\beta})$  with  $\dot{\beta} \in \bar{\mathfrak{H}}_b'' \oplus \mathfrak{H}_b'$  and  $\ddot{x}_0 \in V \oplus (\bar{\mathfrak{H}}_b'' \oplus \mathfrak{H}_b')$ , then

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \Re(R(x(t))) = \int_X (|\dot{\beta}^{01}|^2 - |\dot{\beta}^{10}|^2) \text{vol}_g .$$

Note that, for obtaining the last formula, we used the important cancellation

$$(22) \quad \int \Lambda_g dd^c (Q^{-1} p_r \{ |\dot{\beta}^{01}|^2 - |\dot{\beta}^{10}|^2 \}) \text{vol}_g = 0 ,$$

which follows from the Gauduchon condition. To complete the proof one chooses  $\sigma \in \Sigma_0$  and applies (22) to  $x_t = \iota(\mathfrak{B}(u_t))$ , where  $(u_t)$  is a path in  $\mathfrak{H}_\sigma$  with  $u_0 = 0$ .  $\blacksquare$

#### 4. APPENDIX

**4.1. Hodge type decomposition surfaces on Gauduchon surfaces.** In this section a *complex surface* is a compact, connected 2-dimensional complex manifold. Let  $X$  be a complex surface. We denote by  $A^0(X, \mathbb{R})_r$  (respectively  $A^0(X, \mathbb{C})_r$ ) the kernel of the operator  $\int_X$  on the space  $A^0(X, \mathbb{R})$  (respectively  $A^0(X, \mathbb{C})$ ), i.e., the  $L^2$ -orthogonal complements of the line of constants in this space. We will also denote by the same symbol  $p_r$  the  $L^2$ -orthogonal projections

$$A^0(X, \mathbb{R}) \xrightarrow{p_r} A^0(X, \mathbb{R})_r , \quad A^0(X, i\mathbb{R}) \xrightarrow{p_r} A^0(X, i\mathbb{R})_r , \quad A^0(X, \mathbb{C}) \xrightarrow{p_r} A^0(X, \mathbb{C})_r .$$

The operator

$$Q := p_r \Lambda d^c d|_{A^0(X, \mathbb{R})_r} : A^0(X, \mathbb{R})_r \rightarrow A^0(X, \mathbb{R})_r$$

is an isomorphism (see Proposition 1.2.8 p. 33 [LT1]), and the subspace  $\ker(p_r \Lambda_g d^c)$  of  $A^1(X, \mathbb{R})$  is a topological complement of  $d(A^0(X, \mathbb{R}))$  in  $A^1(X, \mathbb{R})$ .

**Theorem 4.1.** *Let  $(X, g)$  be a Gauduchon surface. Put*

$$H := \{a \in iA^1(X) \mid \bar{\partial}a^{01} = 0, \quad p_r \Lambda_g da = 0, \quad p_r \Lambda_g d^c a = 0\}$$

$$H^{0,1} := \{\alpha \in A^{0,1}(X) \mid \bar{\partial}\alpha = 0, \quad p_r \Lambda_g \partial\alpha = 0\} \subset Z_{\bar{\partial}}^{01}(X) \subset A^{0,1}(X) ,$$

$$H_{\text{cl}} := \{a \in iA^1(X) \mid da = 0, \quad p_r \Lambda_g d^c a = 0\} ,$$

$$H_{\text{cl}}^{0,1} := \{\alpha \in A^{0,1}(X) \mid d(-\bar{\alpha} + \alpha) = 0, \quad p_r \Lambda_g \partial\alpha = 0\} ,$$

$$V^{0,1} := \{\alpha \in A^{0,1}(X) \mid \Lambda_g \partial\alpha = 0\} , \quad V := \{a \in iA^1(X, \mathbb{R}) \mid \Lambda_g da = \Lambda_g d^c a = 0\} .$$

(1) *The natural morphisms*

$$H^{0,1} \rightarrow H_{\bar{\partial}}^{0,1}(X) = H^1(X, \mathcal{O}_X) , \quad H_{\text{cl}} \rightarrow H_{\text{DR}}^1(X, i\mathbb{R})$$

*are isomorphisms.*

(2) *Denoting by  $J$  the obvious complex structure on  $H$ , the map  $a \mapsto a^{01}$  induces a complex isomorphism  $(H, J) \rightarrow H^{0,1}$  which restricts to a real isomorphism  $H_{\text{cl}} \rightarrow H_{\text{cl}}^{0,1}$ .*

(3)  *$H_{\text{cl}}$  is the kernel of the linear functional  $H \rightarrow \mathbb{R}$  defined by  $a \mapsto \int i\omega_g \wedge da$ . This functional is trivial if and only if  $b_1(X)$  is even.*

(4) *Denoting by  $J$  the obvious complex structure on  $V$ , the map  $v \mapsto v^{01}$  induces a complex isomorphism  $(V, J) \rightarrow V^{0,1}$ .*

(5) *When  $b_1(X)$  is even, then  $H^{0,1} \subset V^{0,1}$  and  $H \subset V$ . When  $b_1(X)$  is odd then  $H^{0,1} \cap V^{0,1}$  (respectively  $H \cap V$ ) has complex codimension 1 in  $H^{0,1}$  (respectively in  $(H, J)$ ).*



- (6) If  $b_1(X) = 1$  then  $H^{0,1} \cap V^{0,1} = \{0\}$ ,  $H \cap V = \{0\}$ , and one has direct sum decompositions

$$A^{0,1}(X) = H^{0,1} \oplus V^{0,1} \oplus \bar{\partial}(A^0(X, \mathbb{C})) ,$$

$$iA^1(X, \mathbb{R}) = H \oplus V \oplus d^c(A^0(X, i\mathbb{R})) \oplus d(A^0(X, i\mathbb{R})) .$$

We also mention the following important

**Proposition 4.2.** *Let  $(X, g)$  be a connected Gauduchon compact complex manifold, and let  $(\mathcal{E}, h)$  be a Hermitian holomorphic bundle on  $X$  whose Chern connection  $A$  is Hermite-Einstein with vanishing Einstein constant. Then*

1.  $\ker(d_A) = \ker(\bar{\partial}_A) = H^0(\mathcal{E})$ ,
2. If  $\ker(d_A) = \{0\}$  then
  - (a)  $\ker \Lambda_g \partial_A$  is complement of  $\bar{\partial}_A(A^0(E))$  in  $A^{0,1}(E)$ .
  - (b) The space  $H_A^{0,1} := \{\alpha \in A^{0,1}(E) \mid \bar{\partial}_A(\alpha) = 0, \Lambda_g \partial_A \alpha = 0\}$  is identified with  $H^1(\mathcal{E})$  via the obvious morphism.

*Proof.* Since  $\Lambda_g F_A = 0$  we get  $\Lambda \partial_A \bar{\partial}_A = -\Lambda \bar{\partial}_A \partial_A$ . Using the maximum principle for the operator  $i\Lambda_g \bar{\partial} \partial$  as in the proof of Theorem 2.2.1 p. 50 [LT1] one can prove that  $\ker(\Lambda_g \partial_A \bar{\partial}_A) = \ker d_A$ , which proves 1. If we assume that  $\ker(d_A) = \{0\}$ , then the operator  $\Lambda_g \partial_A \bar{\partial}_A$  will be injective hence, since it has vanishing index, it will be an isomorphism. It follows that for any  $\alpha \in A^{0,1}(E)$  there exists a unique section  $\varphi \in A^0(E)$  such that  $\Lambda_g \partial_A(\alpha + \bar{\partial}_A \varphi) = 0$ . This proves 2. The third statement follows from 2. and the Dolbeault theorem.  $\blacksquare$

**4.2. Integrable semiconnections.** Let  $X$  be a compact connected complex manifold, and  $E$  a differentiable complex vector bundle of rank  $r$  on  $X$ . We recall that a semiconnection on  $L$  is a first order differential operator  $\eta : A^0(E) \rightarrow A^{0,1}(E)$  satisfying the Leibniz rule

$$\eta(\varphi s) = (\bar{\partial} \varphi) \otimes s + \varphi \eta(s) \quad \forall \varphi \in A^0(X, \mathbb{C}) \quad \forall s \in A^0(E) .$$

We denote by  $\mathcal{A}^{0,1}(E)$  the space of semiconnections on  $E$ ; this space has the structure of an affine space with model vector space  $A^{0,1}(\text{End}(E))$ . A semiconnection  $\eta$  on  $E$  admits natural extensions  $A^{0,q}(E) \rightarrow A^{0,q+1}(E)$  satisfying the obvious Leibniz rule, and which will be denoted by the same symbol  $\eta$ . In particular one can consider the composition  $\eta \circ \eta : A^0(E) \rightarrow A^{0,2}(E)$ , which is 0-order operator, hence it is given by multiplication with a form  $F_\eta^{02} \in A^{0,2}(\text{End}(E))$ . A semiconnection  $\eta$  on  $E$  is called integrable if  $F_\eta^{02} = 0$ . If this is the case, then  $\eta$  defines a holomorphic structure  $\mathcal{E}_\eta$  on  $E$ . We will use the same symbol for the corresponding holomorphic vector bundle. The corresponding locally free sheaf on  $X$  is just the sheaf of germs of local sections  $s$  of  $E$  satisfying the equation  $\eta(s) = 0$ . The assignment  $\eta \mapsto \mathcal{E}_\eta$  defines a bijection between the space  $\mathcal{A}^{0,1}(E)^{\text{int}}$  of integrable semiconnections on  $E$  and the space  $\mathcal{H}(E)$  of holomorphic structures on  $E$ . The group  $\Gamma(X, \text{GL}(E))$  acts naturally on  $\mathcal{A}^{0,1}(E)$  by the formula

$$\varphi \cdot \eta := \varphi \circ \eta \circ \varphi^{-1} = \eta - (\eta \varphi) \varphi^{-1} .$$

Fix an integrable semiconnection  $\delta$  on  $D := \det(E)$ , and denote by  $\mathcal{D}$  the corresponding holomorphic structure on  $D$ . The subspace

$$\mathcal{A}_\delta^{0,1}(E) := \{\eta \in \mathcal{A}^{0,1}(E) \mid \det(\eta) = \delta\} \subset \mathcal{A}_\delta^{0,1}(E)$$

is closed and invariant under the action of the gauge group  $\mathcal{G}_E^{\mathbb{C}} := \Gamma(X, \mathrm{SL}(E))$ . The space  $\mathcal{H}_{\mathcal{D}}(E)$  of holomorphic structures on  $E$  which induce  $\mathcal{D}$  on  $\det(E)$  appearing in the definitions of the moduli spaces  $\mathcal{M}_{\mathcal{D}}^{\mathrm{pst}}(E)$ ,  $\mathcal{M}_{\mathcal{D}}^{\mathrm{st}}(E)$ ,  $\mathcal{M}_{\mathcal{D}}^{\mathrm{si}}(E)$  of section 1.2 can be identified with the subspace

$$\mathcal{A}_{\delta}^{0,1}(E)^{\mathrm{int}} := \{\eta \in \mathcal{A}^{0,1}(E)^{\mathrm{int}} \mid \det(\eta) = \delta\} \subset \mathcal{A}^{0,1}(E)^{\mathrm{int}} .$$

#### 4.3. The gauge theoretical Picard group of a compact complex manifold.

Let  $X$  be a compact connected complex  $n$ -manifold, and  $L$  a differentiable complex line bundle on  $X$ . The space  $\mathcal{A}^{0,1}(L)$  of semiconnections on  $L$  is an affine space with model vector space  $A^{0,1}(X)$ , which is independent of  $L$ . The obstruction  $F_{\sigma}$  to integrability of a semiconnection  $\sigma \in \mathcal{A}^{0,1}(L)$  is an element of the space  $A^{0,2}(X)$ , which is also independent of  $L$ . Note that

$$\mathcal{A}^{0,1}(L)^{\mathrm{int}} \neq \emptyset \Leftrightarrow c_1(L) \in \mathrm{NS}(X) ,$$

where  $\mathrm{NS}(X)$  is the kernel of the obvious morphism  $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X)$ .

For a semiconnection  $\sigma \in \mathcal{A}^{0,1}(L)$  and a form  $v \in A^{0,1}(X)$  one has

$$F_{\sigma+v}^{02} = F_{\sigma}^{02} + \bar{\partial}v ,$$

which shows that the subspace  $\mathcal{A}^{0,1}(L)^{\mathrm{int}} \subset \mathcal{A}^{0,1}(L)$  of integrable semiconnections on  $L$ , if *non-empty*, is an affine subspace of  $\mathcal{A}^{0,1}(L)$  whose model vector space is the space  $Z_{\bar{\partial}}^{0,1}(X)$  of  $\bar{\partial}$ -closed  $(0,1)$ -forms. We will denote by  $\mathcal{L}_{\sigma}$  the holomorphic structure on  $L$  (and the holomorphic line bundle) defined by an integrable semiconnection  $\sigma$ . The natural action of the complex gauge group  $\mathcal{G}^{\mathbb{C}} := \mathcal{C}^{\infty}(X, \mathbb{C})$  on  $\mathcal{A}^{0,1}(L)$  is given by the formula

$$\varphi \cdot \sigma := \varphi \circ \sigma \circ \varphi^{-1} = \sigma - (\bar{\partial}\varphi)\varphi^{-1} .$$

and the stabilizer of any point  $\sigma \in \mathcal{A}^{0,1}(L)$  with respect to this action is  $\mathbb{C}^*$ , so we obtain an induced free action of the reduced complex gauge group  $\mathcal{G}_0^{\mathbb{C}} := \mathcal{G}^{\mathbb{C}}/\mathbb{C}^*$ . This gauge action leaves invariant the closed subspace  $\mathcal{A}^{0,1}(L)^{\mathrm{int}}$ , and, assuming  $c_1(L) \in \mathrm{NS}(X)$ , the quotient

$$\mathcal{M}(L) := \mathcal{A}^{0,1}(L)^{\mathrm{int}} / \mathcal{G}^{\mathbb{C}}$$

is a  $h^1(\mathcal{O}_X)$ -dimensional dimensional complex manifold which can be identified with the connected component  $\mathrm{Pic}^{c_1(L)}(X)$  of the Picard group  $\mathrm{Pic}(X)$ . This manifold can be regarded as a subspace of the infinite dimensional quotient

$$\mathcal{B}^{0,1}(L) := \mathcal{A}^{0,1}(L) / \mathcal{G}^{\mathbb{C}} .$$

For a fixed point  $x_0 \in X$  denote by  $\mathcal{G}_{x_0}^{\mathbb{C}}$  the kernel of the evaluation morphism  $\mathrm{ev}_{x_0} : \mathcal{G}^{\mathbb{C}} \rightarrow \mathbb{C}^*$ . This group is naturally isomorphic to  $\mathcal{G}_0^{\mathbb{C}}$  and acts freely on  $\mathcal{A}(L)$ . The quotient

$$\mathcal{A}^{0,1}(L)^{\mathrm{int}} \times_{\mathcal{G}_{x_0}^{\mathbb{C}}} L$$

can be naturally regarded as a line bundle on the product  $\mathcal{M}(L) \times X$ . This line bundle comes with a tautological holomorphic structure and will be denoted by  $\mathcal{L}_{x_0}$ . Via the identification  $\mathcal{M}(L) = \mathrm{Pic}^{c_1(L)}(X)$  this line bundle corresponds to the Poincaré line bundle normalized at  $x_0$ .

Endow  $L$  with a Hermitian metric  $h$ . The gauge group of the Hermitian line bundle  $L$  is  $\mathcal{G} := \mathcal{C}^\infty(X, S^1)$ . We recall that a Hermitian connection  $b$  on  $(L, h)$  is called Hermitian-Einstein if it satisfies the equations

$$F_b^{02} = 0, \quad p_r[\Lambda_g F_b] = 0.$$

Note that the second condition  $p_r[\Lambda_g F_b] = 0$  is equivalent to the classical Hermitian-Einstein condition “ $i\Lambda_g F_b$  is a constant” (called the Einstein constant of the connection). In the non-Kählerian framework the map which assigns to a holomorphic line bundle the Einstein constant of a compatible Hermite-Einstein connection is not necessarily constant on  $\text{Pic}^c(X)$ , because the degree map associated with a Gauduchon metric is not a topological invariant in general [LT1].

The space  $\mathcal{A}^{\text{HE}}(L) \subset \mathcal{A}(L)$  of Hermite-Einstein connections on  $L$  is an affine subspace with model vector space

$$\mathcal{H} := \{a \in A^1(X, i\mathbb{R}) \mid \bar{\partial}a^{01} = 0, \quad p_r\Lambda da = 0\}.$$

Therefore, fixing a Hermite-Einstein connection  $b_0 \in \mathcal{A}^{\text{HE}}(L)$ , we have

$$\mathcal{A}^{\text{HE}}(L) = b_0 + \mathcal{H}.$$

As shown in [LT1] in gauge theory it is convenient to replace the usual “Coulomb slice condition”  $d^*a = 0$  on 1-forms by the condition  $p_r\Lambda_g d^c a = 0$ . The two conditions are equivalent in the Kählerian case; in the general Gauduchon case they both define slices for the action of  $\mathcal{G}$  on  $\mathcal{A}(L)$ . The advantage of the new slice condition introduced in [LT1] is that the intersection

$$H := \{a \in iA^1(X) \mid \bar{\partial}a^{01} = 0, \quad p_r\Lambda_g da = 0, \quad p_r\Lambda_g d^c a = 0\}$$

of  $\mathcal{H}$  with  $\ker(p_r\Lambda_g d^c)$  is  $J$ -invariant, so comes with a natural complex structure. We define the subgroups

$$G := \{\varphi \in \mathcal{G} \mid p_r\Lambda d^c(\varphi^{-1}d\varphi) = 0\}, \quad G^{\mathbb{C}} := \{\varphi \in \mathcal{G}^{\mathbb{C}} \mid p_r\Lambda_g \partial(\varphi^{-1}\bar{\partial}\varphi) = 0\}$$

of  $\mathcal{G}$  and  $\mathcal{G}^{\mathbb{C}}$  respectively.

**Proposition 4.3.** *Let  $X$  be a compact complex surfaces endowed with a Gauduchon metric  $g$ .*

(1) *One has  $G^{\mathbb{C}} = G \times \mathbb{R}_{>0}$ .*

(2) *The map  $\mathcal{G}^{\mathbb{C}} \rightarrow H^1(X, \mathbb{C})$  given by  $\varphi \mapsto [\varphi^{-1}d\varphi]_{\text{DR}}$  takes values in the group  $2\pi i H^1(X, \mathbb{Z})$  and induces*

(a) *An epimorphism  $q : G \rightarrow 2\pi i H^1(X, \mathbb{Z})$  and a short exact sequence*

$$\{1\} \rightarrow S^1 \rightarrow G \rightarrow 2\pi i H^1(X, \mathbb{Z}) \rightarrow \{1\},$$

(b) *An epimorphism  $q^{\mathbb{C}} : G^{\mathbb{C}} \rightarrow 2\pi i H^1(X, \mathbb{Z})$  and a short exact sequence*

$$\{1\} \rightarrow \mathbb{C}^* \rightarrow G^{\mathbb{C}} \rightarrow 2\pi i H^1(X, \mathbb{Z}) \rightarrow \{1\}.$$

*Proof.* (1) For  $\varphi \in \mathcal{G}^{\mathbb{C}}$  write locally  $\varphi = e^f$  for a (locally defined) smooth complex function  $f$  and note that

$$\partial(\varphi^{-1}\bar{\partial}\varphi) = \partial\bar{\partial}f, \quad \partial(\bar{\varphi}^{-1}\bar{\partial}\bar{\varphi}) = \partial\bar{\partial}\bar{f}, \quad \partial((\varphi\bar{\varphi})^{-1}\bar{\partial}(\varphi\bar{\varphi})) = \partial\bar{\partial}(f + \bar{f}).$$

Since  $i\partial\bar{\partial}$ , these formulae show that

$$p_r\Lambda_g \partial(\varphi^{-1}\bar{\partial}\varphi) = 0 \Rightarrow p_r\Lambda_g \partial((\varphi\bar{\varphi})^{-1}\bar{\partial}(\varphi\bar{\varphi})) = 0 \Rightarrow p_r\Lambda_g d^c d \log |\varphi|^2 = 0,$$

which implies that  $|\varphi|$  is constant. Therefore any element  $\varphi \in G^{\mathbb{C}}$  can be written as  $\varphi = ce^{\psi}$  where  $c \in \mathbb{R}_{>0}$  and  $\psi \in \mathcal{G}$ . Writing locally  $\psi = e^g$  for a pure imaginary

(locally defined) function we get  $\Lambda_g \partial \bar{\partial} g = 0$ , which is equivalent to  $\Lambda_g d^c dg = 0$ . This implies  $\Lambda_g d^c(\psi^{-1} d\psi) = 0$ , hence  $\psi \in G$ .

(2) The Cauchy formula shows that the Rham cohomology class of the complex 1-form  $\frac{1}{2\pi i} z^{-1} dz$  is the canonical generator  $\gamma$  of  $H^1(\mathbb{C}^*, \mathbb{Z})$ . Therefore for any  $\varphi \in \mathcal{G}^{\mathbb{C}}$  one has

$$[\varphi^{-1} d\varphi]_{\text{DR}} = 2\pi i \varphi^*(\gamma) \in 2\pi i H^1(X, \mathbb{Z}) .$$

(a) Since  $S^1$  is a  $K(\mathbb{Z}, 1)$ -space it follows easily that the map

$$[X, S^1] \rightarrow H^1(X, \mathbb{Z})$$

given by  $[\varphi] \mapsto \varphi^*(\gamma)$  is an isomorphism. Therefore for every  $y \in 2\pi i H^1(X, \mathbb{Z})$  there exists  $\varphi_y \in \mathcal{G}$  with  $[\varphi_y^{-1} d\varphi_y]_{\text{DR}} = y$ . It suffices to find  $f \in iA^0(X, \mathbb{R})$  such that putting  $\varphi = e^f \varphi_y$  one has  $p_r \Lambda_g d^c(\varphi^{-1} d\varphi) = 0$ . This condition is equivalent to  $p_r \Lambda_g d^c df + p_r \Lambda_g d^c(\varphi_y^{-1} d\varphi_y) = 0$ . Since the operator  $Q$  is an isomorphism, this equation has a unique solution  $f \in iA^0(X, \mathbb{R})_r$ .

(b) This follows from (1) and (2) (a). ■

**Corollary 4.4.** *Denoting  $G_{x_0} := \ker(ev_{x_0} : G \rightarrow S^1)$ ,  $G_{x_0}^{\mathbb{C}} := \ker(ev_{x_0} : G^{\mathbb{C}} \rightarrow \mathbb{C}^*)$  one has*

$$G_{x_0}^{\mathbb{C}} = G_{x_0} = \{\varphi \in \mathcal{C}^\infty(X, S^1) \mid p_r \Lambda_g d^c(\varphi^{-1} d\varphi) = 0, \varphi(x_0) = 1\} .$$

**Corollary 4.5.** (1) *Let  $b_0$  be a Hermite-Einstein connection on  $L$ . The embedding  $b_0 + H \hookrightarrow b_0 + \mathcal{H}$  induces an isomorphism of real analytic moduli spaces*

$$b_0 + H /_G \xrightarrow{\cong} \mathcal{M}^{HE}(L) .$$

(2) *Let  $\sigma_0$  be an integrable semiconnection on  $L$ . The embedding  $\sigma_0 + H^{0,1} \hookrightarrow \sigma_0 + \mathcal{Z}_{\bar{\partial}}^{0,1}(X)$  induces an isomorphism of complex moduli spaces*

$$\sigma_0 + H^{0,1} /_{G^{\mathbb{C}}} \xrightarrow{\cong} \mathcal{M}(L) .$$

The stabilizer of a point  $b = b_0 + h \in b_0 + H$  (respectively of a point  $\sigma = \sigma_0 + \chi \in \sigma_0 + H^{0,1}$ ) with respect to the  $G$ -action (respectively  $G^{\mathbb{C}}$ -action) is  $S^1$  (respectively  $\mathbb{C}^*$ ). Therefore, using Proposition 4.3, we obtain the following finite dimensional descriptions of the moduli spaces:

**Remark 4.6.** *We have natural identifications*

$$(23) \quad \mathcal{M}^{HE}(L) = b_0 + H /_{2\pi i H^1(X, \mathbb{Z})} , \quad \mathcal{M}(L) = \sigma_0 + H^{0,1} /_{2\pi i H^1(X, \mathbb{Z})} ,$$

where  $2\pi i H^1(X, \mathbb{Z})$  acts on the two affine spaces via the identifications

$$G /_{S^1} = 2\pi i H^1(X, \mathbb{Z}) , \quad G^{\mathbb{C}} /_{\mathbb{C}^*} = 2\pi i H^1(X, \mathbb{Z})$$

induced by the epimorphisms  $q, q^{\mathbb{C}}$ . Choosing  $\sigma_0 = \bar{\partial}_{b_0}$ , the Kobayashi-Hitchin isomorphism

$$\mathcal{M}^{HE}(L) \xrightarrow{\simeq KH} \mathcal{M}(L)$$

is induced via the formulae (23) by the isomorphism  $I : H \rightarrow H^{0,1}$ .

Put  $\Sigma := \sigma_0 + H^{0,1}$ . The product  $\Sigma \times L$  can be regarded as a line bundle over  $\Sigma \times X$ . This line bundle comes with a tautological integrable semiconnection  $\sigma_{\text{taut}}$  characterized by the following conditions:

- (1) For any  $\sigma \in \Sigma$  the restriction of  $\sigma_{\text{taut}}$  to the bundle  $\{\sigma\} \times L$  over the fiber  $\{\sigma\} \times X$  coincides with  $\sigma$  via the obvious identifications.
- (2) For any  $x \in X$  the restriction of  $\sigma_{\text{taut}}$  to the line bundle  $\Sigma \times L_x$  over the slice  $\Sigma \times \{x\}$  coincides with the standard trivial semiconnection on this trivial line bundle.

Endowing  $\Sigma \times L$  with the obvious product  $G^{\mathbb{C}}$ -action, we see that  $\sigma_{\text{taut}}$  is  $G^{\mathbb{C}}$ -invariant. Fixing  $x_0 \in X$  we can regard  $\Sigma \times X$  as a principal  $G_{x_0}$ -bundle over the product  $\mathcal{M}(L) \times X$ , hence we can construct the associated vector bundle

$$(\Sigma \times X) \times_{G_{x_0}^{\mathbb{C}}} L = \Sigma \times L / G_{x_0},$$

which will be regarded as a line bundle over  $\mathcal{M}(L) \times X$ .

**Definition 4.7.** *The universal (Poincaré) line bundle normalized at a point  $x_0 \in X$  is the holomorphic bundle  $\mathcal{L}_{x_0}$  obtained by endowing the quotient bundle  $\Sigma \times L / G_{x_0}$  over  $\mathcal{M}(L) \times X$  with the integrable semiconnection induced by  $\sigma_{\text{taut}}$*

**Remark 4.8.** *Via the standard identification  $\mathcal{M}(L) = \text{Pic}^{c_1(L)}(X)$ , the universal bundle  $\mathcal{L}_{x_0}$  coincides with the Poincaré line bundle normalized at  $x_0$ .*

In a similar way, putting  $S := b_0 + H$  one obtains a universal Hermitian connection  $\mathbb{A}_{x_0}$  on the universal Hermitian line bundle  $\mathbb{L}_{x_0} = S \times L / G_{x_0}$ .

**Remark 4.9.** *Since  $G_{x_0}^{\mathbb{C}} = G_{x_0}$ , the former group also acts by unitary isomorphisms on the line bundle  $\Sigma \times L$ , hence we obtain a Hermitian metric on the universal holomorphic line bundle  $\mathcal{L}_{x_0}$ . This metric is fiberwise (in the  $X$ -direction) Hermitian-Einstein, and depends only on  $g$  and the metric  $h_{x_0}$  on the line  $L_{x_0}$ . It can be obtained solving fiberwise the Hermitian-Einstein equation.*

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